Chapter 1: Real numbers

Problem 1. Negate the following statements (don't pay attention to the correctness of the statements):

- Every time it rains, the grass gets wet.

- All cats are friendly.

- No birds can swim.

- On occasion, the sky turns brown.

- There is an integer n such that $n^2 = 2$.

- If n is an integer, then n^2 is also an integer.

Problem 2 (Proof by cases). If n is a natural number, then $n^2 + n$ is even.

Problem 3 (Direct proof). Prove the following statement: Let n be a natural number. If n is odd, then n^2 is odd.

Problem 4 (Contrapositive proof). Prove the following statement: Let n be a natural number. If n^2 is odd, then n is odd.

Problem 5 (Proof by contradiction). Prove that there is no real number r such that $r^2 = 2$.

Problem 6 (Quantifiers). Consider the following statements:

(a) $\exists n \in \mathbb{Z}, \forall m \in \mathbb{Z}, m + n > 0$.

(b) $\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}, m + n > 0$.

(c) $\exists n \in \mathbb{Z}, \exists m \in \mathbb{Z}, m + n > 0$.

(d) $\forall n \in \mathbb{Z}, \forall m \in \mathbb{Z}, m + n > 0$.

$\mathbb{Z}$ is the set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. It comes from the German word *Zahlen*, which means *numbers*.

Problem 7 (Set inclusion). Let $A = \{n^2 \mid n \in \mathbb{N}\}$ and $B = \{n^4 \mid n \in \mathbb{N}\}$. Show that $B \subset A$.

Problem 8 (Set equality). Let A, B and C be sets. Show that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Problem 9. Given two distinct rational numbers p and q, show that there is a rational number r such that $\min(p, q) < r < \max(p, q)$. Does the same hold if we replace *rational number* with *integer*?

Problem 10. Given any rational number p such that $p^2 < 2$, there is another rational number q such that $q^2 < 2$ and $q > p$.

In the book, the following choice $q = p - \frac{p^2-2}{p+2}$ is proposed. We will discuss the geometric idea behind it, but you should think about coming up with your own choice.

Problem 11 (Proposition 1.15). Show that the axioms for multiplication imply the following statements.

(a) If $x \neq 0$ and $xy = xz$ then $y = z$.

(b) If $x \neq 0$ and $xy = x$ then $y = 1$.

(c) If $x \neq 0$ and $xy = 1$ then $y = 1/x$.

(d) If $x \neq 0$ then $1/(1/x) = x$.

Problem 12. Let S be an ordered set, and let $E \subset S$. Assume that there is $\alpha \in E$ such that α is an upper bound of E. Show that

$$\alpha = \sup E.$$

In this situation, α is usually called the maximum of E.

Problem 13 (Problem 1.7, part (a)). Let $b > 1$ and let n be a positive integer. Show that
$$b^n - 1 \geq n(b - 1)$$

This is called Bernoulli's inequality.

Problem 14 (Theorem 1.35). Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathbf{C}$, show that
$$\left| \sum_{i=1}^n a_i \overline{b_i} \right| \leq \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2} \left(\sum_{i=1}^n |b_i|^2 \right)^{1/2}$$

This is known as the Cauchy-Schwarz inequality

Problem 15 (Problem 1.5). Let A be a non-empty set of real numbers which is bounded below. Prove that

$$\inf(A) = -\sup(-A),$$

where $-A = \{-x \mid x \in A\}$.

Problem 16 (Problem 1.20). Prove that no order can be defined in the complex field that turns it into an ordered field. Hint: -1 is a square.

Problem 17. In the lecture, we will go through the definition (and existence) of the n-th root of a real number using the LUP. Our next step is to define the real power of a real number, and the logarithm. Basically, we will work through problems 1.6 and 1.7.

Chapter 2: Basic Topology

Problem 18 (Set equality). Let A, B and C be sets. Show that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Hint: To prove $X = Y$, you can prove that $X \subset Y$ and $Y \subset X$.

Problem 19. Let A and B be two countable sets. Show that $A \cup B$ is countable.

Problem 20. Let A, B, C be sets. Prove the following:

1. $A \sim A$.

2. If $A \sim B$, then $B \sim A$.[1]

3. If $A \sim B$ and $B \sim C$, then $A \sim C$.

[1] Basically, you need to show that if there a 1-1 and onto map $f : A \to B$, there is another 1-1 and onto map $g : B \to A$.

Problem 21. Show that the set of irrational numbers is uncountable.

Problem 22. Let A be a non-empty set and let $\mathcal{P}(A)$ be the power set of A, *i.e.* the set of all sets of A.

Show that there is no onto map f from A to $\mathcal{P}(A)$.[2]

Hint: Consider the set $\{x \in A \mid x \notin f(x)\}$.

[2] This is known as Cantor's theorem. It tells us that given any set, there is a *set*.

Problem 23. Show that the countable union of finite sets is countable.[3]

[3] In fact, the countable union of countable sets is countable, but the proof is a bit more involved (read more here).

Problem 24. Let $\mathbb{A}$ the set of all roots of polynomials with integer coefficients[4]. Show that $\mathbb{A}$ is countable.

[4] $\mathbb{A}$ is called the set of algebraic numbers

Problem 25 (Problem 2.11). For $x, y \in \mathbb{R}$, define

$$d_1(x, y) = (x - y)^2,$$
$$d_2(x, y) = |x - y|^{1/2},$$
$$d_3(x, y) = |x^2 - y^2|,$$
$$d_4(x, y) = |x - 2y|,$$
$$d_5(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

Determine, for each of these, whether it is a metric or not.

Problem 26. Let d_1, d_2 be two metrics on a metric space S. Show that $d_1 + d_2$ and $\max(d_1, d_2)$ are also metric on S.[5]

[5] $\max(d_1, d_2)$ is the metric that associates with (x, y), the maximum between $d_1(x, y)$ and $d_2(x, y)$.

Problem 27 (Nested interval theorem). Let $(I_n)_{n=1}^{\infty}$ be a sequence of closed and bounded intervals[6] such that $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$. Show that the intersection

$$\bigcap_{n=1}^{\infty} I_n$$

is not empty.

[6] This is Theorem 2.38 in the book. Note that Rudin uses "*intervals*" to mean "*closed and bounded intervals*"

Problem 28. Let $(K_n)_n$ be a sequence of non-empty compact sets such that $K_{n+1} \subset K_n$. Show that the intersection

$$\bigcap_{n=1}^{\infty} K_n$$

is not empty.

Problem 29. Does the nested interval theorem hold for open intervals? That is, must the intersection of a sequence of nested non-empty open intervals be non-empty? Same question for closed sets.

Problem 30 (Rudin 2.16). Let $A = \{r \in \mathbb{Q} \mid r > 0, 2 < r^2 < 3\}$. Show that A is bounded, closed in $\mathbb{Q}$ (with the usual metric), but it is not compact.[7]

[7] This shows that closed and bounded sets in $\mathbb{Q}$ may not be compact (unlike $\mathbb{R}$).

Problem 31 (Theorem 2.35). Show that closed subsets of compact sets are compact.

Problem 32 (Problem 2.22). Show that there is a countable dense subset of $\mathbb{R}^n$.

Problem 33 (Rudin 2.7). Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

(a) If $B = \cup_{i=1}^{\infty} A_i$, prove that

$$\overline{B} \supset \bigcup_{i=1}^{\infty} \overline{A_i}.$$

Can we replace $\supset$ with $=$ in (a)?

(b) If $B_n = \cup_{i=1}^{n} A_i$, prove that

$$\overline{B_n} = \bigcup_{i=1}^{n} \overline{A_i},$$

for $n \in \mathbb{N}$.

Problem 34. Let $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Show that A is not compact, and that $A \cup \{0\}$ is compact.

Chapter 3: Sequences and Series

Problem 35. Let $(s_n)_n$ be a real sequence and let $m \in \mathbb{N}$. Show that if $s_n \to s$, then $s_n^m \to s_n^m$.

Problem 36 (Rudin 3.1)**.** Let $(s_n)_n$ be a real sequence[8], Show that if $(s_n)_n$ converges, then $(|s_n|)_n$ converges too. Is the converse true?

[8] The same is true for complex sequences

Problem 37. Show that the sequence $(\sin(n))_n$ does not converge.[9] *Hint:* There are many ways to solve this problem. Here are two:

[9] In fact, this sequence is dense in $[-1, 1]$.

- Assume that the sequence converges, show that $(\cos(n))_n$ will also converge using $\sin(n + 1) = \dots$.

- Show that this would lead to $\cos(1) = 1$. This approach requires knowing that $\cos(1) \in (0, 1)$.

Another approach is to use the trig circle to show that $\sin(n) > \frac{1}{2}$ for infinitely many n's and $\sin(n) < \frac{-1}{2}$ for infinitely many n's too.

Problem 38. Let $(a_n)_n$ be a real sequence such that $(na_n)_n$ converges. Show that $a_n \to 0$.
Question: Can we generalize this more by replacing n with an unbounded real/complex sequence?
Question: Can we generalize this to vector valued sequences by replacing the scalar multiplication with a dot product: If $(\mathbf{a}_n \cdot \mathbf{b}_n)_n$ converges and $\mathbf{b}_n$ is unbounded, does it follow that $\mathbf{a}_n \to \mathbf{0}$?

Problem 39 (Squeeze theorem)**.** Let $(a_n)_n$, $(b_n)_n$ and $(c_n)_n$ be three real sequences such that

$$a_n \leq b_n \leq c_n, \qquad \forall n \in \mathbb{N},$$

and the sequences $(a_n)_n$ and $(c_n)_n$ converge to the same number l. Prove that $b_n \to l$.

Problem 40. Let $E \subset \mathbb{R}$ be a non-empty bounded-above set. Show that there is a sequence $(a_n)_n$ in E such that $a_n \to \sup E$.

Problem 41. Let $q \in (0, 1)$. Show that $q^n \to 0$.
Hint: Use Bernoulli's inequality: $(1 + x)^n \geq 1 + nx$ for $x \geq 0$.

Problem 42. Let $(a_n)_n$ be a real sequence that there is $C > 0$, and $q \in (0, 1)$ for which

$$|a_{n+1} - a_n| < Cq^n, \qquad \forall n \in \mathbb{N}.$$

Show that $(a_n)_n$ is a Cauchy sequence.

Problem 43 (Euler's number)**.** Let $(a_n)_n$ be the sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

- Show that $(a_n)_n$ is bounded above. *Hint:* Use the binomial theorem.

- Show that $(a_n)_n$ is increasing. *Hint:* Show that $\frac{a_{n+1}}{a_n} \geq 1$ using Bernoulli's inequality.

- Deduce that $(a_n)_n$ converges to some real number $e \in \mathbb{R}$.[10]

[10] This limit comes from the problem of continuous compounding interest stdudied by Jacob Bernoulli in 1683.

Problem 44 (Heron's method). Let $(a_n)_n$ be the sequence defined by $a_1 = 1$ and

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right), \qquad n \in \mathbb{N}.$$

- Show that $a_n \geq \sqrt{2}$ for all $n \geq 2$. (*Hint:* : Induction)

- Show that $a_{n+1} - \sqrt{2} \leq \frac{1}{2\sqrt{2}}(a_n - \sqrt{2})^2$.

- Show that $a_n \to \sqrt{2}$.[11]

[11] This is a very fast algorithm for approximating $\sqrt{2}$.

Problem 45 (sum of lim sup). Let $(a_n)_n$ and $(b_n)_n$ be two sequences. Show that

$$\limsup_n (a_n + b_n) \leq \limsup_n a_n + \limsup_n b_n$$

Does the equality hold in general?

Problem 46 (Root test vs ratio test). Let $(a_n)_n$ be real positive sequence. Show that[12]

$$\liminf_n \frac{a_{n+1}}{a_n} \leq \liminf_n \sqrt[n]{a_n} \leq \limsup_n \sqrt[n]{a_n} \leq \limsup_n \frac{a_{n+1}}{a_n}$$

[12] This is a special case of the more general Stolz–Cesàro theorem, whic is a version of L'Hôpital's rule for sequences.

Note: This proves that if the limit of the ratio a_{n+1}/a_n exists, then $\sqrt[n]{a_n}$ will converge to the same limit.

Problem 47. Show that

$$\frac{n}{\sqrt[n]{n!}} \to e,$$

where e is Euler's number. *Hint:* Use Problem 46.

Problem 48 (A special case of Banach fixed point theorem). Let $f : \mathbb{R} \to \mathbb{R}$ be a function for which there is a real number $q \in (0,1)$ such that[13]

$$|f(x) - f(y)| \leq q|x - y|, \qquad \forall x, y \in \mathbb{R}.$$

[13] A such function f is called a contraction.

Let $(a_n)_n$ be the sequence:

$$\begin{cases} a_1 \in \mathbb{R}, \\ a_{n+1} = f(a_n). \end{cases}$$

Show that $(a_n)_n$ converges. *Hint:* Use Problem 42.

Problem 49. Determine whether the following sequences converge or diverge:

(a) $\sum_n \dfrac{n^2}{2^n}$

(d) $\sum_n \dfrac{n^2 + 3n - 1}{n^3 + 2n^2 + 1}$

(b) $\sum_n \dfrac{n^\alpha}{n!}$ where $\alpha \in \mathbb{R}$.

(e) $\sum_n \left(1 + \dfrac{1}{n}\right)^{n^2}$

(c) $\sum_n \dfrac{n^2 + 3n - 1}{n^4 + 2n^2 + 1}$

(f) $\sum_n \left(1 - \dfrac{1}{n}\right)^{n}$

Problem 50. Let $\{f_n\}$ be the Fibonacci sequence[14] defined as:

$$f_1 = f_2 = 1, \ f_{n+2} = f_{n+1} + f_n, \ \forall n \geq 3.$$

The goal of this exercise is to show that

$$\sum_n \frac{1}{f_n} \quad \text{converges.} \tag{1}$$

(a) Let $a_n = \dfrac{f_{n+1}}{f_n}$. Come up with a recurrence for $(a_n)_n$.

(b) Prove that a_n converge to $\dfrac{1 + \sqrt{5}}{2}$.[15]

(c) Conclude that the series in (1) converge.

Problem 51 (Olivier's theorem). Let $(a_n)_n$ be a positive and decreasing sequence. Assume that

$$\sum_n a_n \text{ converges.}$$

Prove that $na_n \to 0$.[16]

Problem 52. Find the domain of convergence of the following power series:

(a) $\sum_n \dfrac{(-1)^n}{(2n)!} x^n$

(c) $\sum_n \dfrac{n!}{n^n} x^n$

(b) $\sum_n \dfrac{(-1)^n}{(2n + 1)} x^n$

(d) $\sum_n \dfrac{1}{\sqrt{n}} x^n$

[14] This sequence appeared in Fibonacci's work on the growth of the rabbit population in 1202, but there are sources discussing this sequence in India much earlier

[15] The number $\frac{1+\sqrt{5}}{2}$ is called the golden ratio.

[16] This results appeared in Crelle in 1827, but not much is known about its author Louis Olivier. It could have been a pseudonym.

Chapter 4: Continuity

Problem 53. Show that $x \mapsto x^2$ is continuous on $\mathbb{R}$. Give a proof using the ϵ-δ definition of continuity, and another proof using the sequential characterization of continuity.[17]

Problem 54. Let $f : X \to \mathbb{R}$ be a continuous function. Show that $|f|$ is continuous.[18]

Is the converse true? That is, if $|f|$ is continuous, does it follow that f is continuous?

Problem 55. Let X, Y and Z be metric spaces, and let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Prove that the composition $g \circ f : X \to Z$ is continuous.

Problem 56. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies

$$\lim_{h \to 0} f(x + h) - f(x - h) = 0, \qquad \text{for every } x \in \mathbb{R}.$$

Does this imply that f is continuous?[19]

Problem 57. If f is a continuous mapping of a metric space X into a metric space Y, prove that

$$f(\overline{E}) \subset \overline{f(E)}$$

for every set $E \subset X$ where $\overline{E}$ denotes the closure of E.[20]

Problem 58. Let f be a continuous real valued[21] function on a metric space X. Let $Z(f)$ be the zero set of f, that is, the set of all $p \in X$ such that $f(p) = 0$. Prove that $Z(f)$ is closed.

Problem 59. If E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{z \in E} d(x, z).$$

(*a*) Prove that $\rho_E(x) = 0$ if and only if $x \in \overline{E}$.
(*b*) Prove that ρ_E is a uniformly continuous function on X by showing that

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y).$$

Problem 60. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$

Determine, with proof, the points at which f is continuous.

[17] Here, $x \mapsto x^2$ is a short hand for "the function that maps x to x^2."

[18] Here, $|f|$ is the function defined by $|f|(x) = |f(x)|$.

[19] If you think the answer is yes, prove it. If you think the answer is no, give a counterexample.

[20] Bonus: Show, by an example, that the inclusion can be proper, that is, $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.
[21] Real valued means that the codomain of f is $\mathbb{R}$. We also, say f is a real function.

Problem 61. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by[22]

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$

Show that f is not continuous at any point of $\mathbb{R}$.

Problem 62. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by[23]

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p \in \mathbb{Z}, q \in \mathbb{N}, \quad \gcd(p, q) = 1 \end{cases}.$$

Show that f is continuous at every irrational point, and that f is not continuous at any rational point.

[22] This is called the Dirichlet function.

[23] This is called Thomae's function.

Chapter 5: Differentiation

Problem 63. Let $f(x) = |x|$ on $\mathbb{R}$. Show that f is not differentiable at zero.[24]

[24] This (combined with theorem 5.2) shows that differentiability is a stronger condition that continuity

Problem 64. You are given that the derivative of sin is cos and the derivative of $x \mapsto x^{-1}$ is $x \mapsto -x^{-2}$. Let

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Prove that f is differentiable everywhere, but that f' is not continuous at zero.[25]

[25] This shows that even if f is differentiable everywhere, f' need not be continuous.

Problem 65 (Rudin 5.12). Let $f : [a, b] \to \mathbb{R}$ be a differentiable function, and suppose that $f'(a) \leq \lambda \leq f'(b)$. Prove that there exists some $c \in (a, b)$ such that $f'(c) = \lambda$.[26]

[26] This shows that the derivative of a function has the *intermediate value property*, even though it need not be continuous.

Problem 66 (Rudin Problem 5.5). Let $f : [0, \infty) \to \mathbb{R}$ be a differentiable function such that $f'(x) \to 0$ as $x \to \infty$. Prove that $g(x) = f(x+1) - f(x)$ converges to zero as $x \to \infty$.

Problem 67. Come up with a formula for the k-th derivative of the product of two functions.[27]

[27] This is called Leibniz's rule.

Problem 68. Let f be a differentiable function at a. Find the following limit

$$\lim_{x \to a} \frac{a^n f(x) - x^n f(a)}{x - a}.$$

Problem 69. Let f be a continuously differentiable function[28] on $[a, b]$. Assume that f has a zero c in (a, b), i.e., $f(c) = 0$. Furthermore, assume that $f'(c) \neq 0$.

[28] "Continuously differentiable means that f' is continuous

We define a sequence via the following iteration:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0,$$

where x_1 is some point in (a, b). Show that if x_1 is sufficiently close to c, then the sequence $\{x_n\}$ converges to c.[29]
Hint: Let $g(x) = x - \frac{f(x)}{f'(x)}$. Show that g is contractive mapping in a neighborhood of c, and use a result from Chapter 3.A.

[29] This is called Newton's method for approximating roots of f.

Problem 70 (Rudin 5.9). Let $f, g : [a, b] \to \mathbb{R}$ be two functions that are continuous on $[a, b]$ and differentiable on (a, b). Prove that there is some $c \in (a, b)$ such that[30]

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

[30] This is called Cauchy's Mean Value Theorem.

Chapter 6: Riemann–Stieltjes Integration

Assumptions and notation: Throughout this recitation, we assume that $\alpha : [a, b] \to \mathbb{R}$ is an increasing function. We denote by $\mathfrak{R}(\alpha)$ the set of all functions $f : [a, b] \to \mathbb{R}$ that are Riemann–Stieltjes integrable with respect to α on $[a, b]$. When $\alpha(x) = x$, we simply write $\mathfrak{R}$ instead of $\mathfrak{R}(\alpha)$ to denote the set of Riemann integrable functions on $[a, b]$.

If the interval is not specified, we assume that the domain of definition of the functions is $[a, b]$.

Problem 71 (Rudin problem 6.1). Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$ f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, \end{cases} $$

and assume that α is continuous at 0. Show that $f \in \mathfrak{R}(\alpha)$.

Problem 72. Let $f(x) = x$. Show that $\int_0^1 f dx = \frac{1}{2}$ without using FTC.

Problem 73 (Rudin problem 6.4). Let $f : [0, 1] \to \mathbb{R}$ be the Dirichlet function defined by

$$ f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in [0, 1]. \end{cases} $$

Show that $f \notin \mathfrak{R}$.

Problem 74 (Rudin problem 6.6). Let f be a bounded function on $[a, b]$ and suppose that $f^2 \in \mathfrak{R}$. Does it follow that $f \in \mathfrak{R}$? Does the answer change if we assume that $f^3 \in \mathfrak{R}$?

> f^2 is the square of f. That is, $f^2(x) = (f(x))^2$. Same for f^3.

Problem 75 (Rudin, parts of Problem 6.10). Let $p, q > 1$ such that

$$ \frac{1}{p} + \frac{1}{q} = 1. $$

(a) (Young's inequality for products) If $u \geq 0$ and $v \geq 0$, then

$$ uv \leq \frac{u^p}{p} + \frac{v^q}{q}. $$

Equality holds if and only if $u^p = v^q$.[31]

(b) If $f, g \in \mathfrak{R}(\alpha)$, $f, g \geq 0$, and

$$ \int_a^b f^p \, d\alpha = 1 = \int_a^b g^q \, d\alpha, $$

then

$$ \int_a^b fg \, d\alpha \leq 1. $$

[31] It stuck on the proof, try to adjust what we did in lectures for proofs: move everything to one side and define that to be a function of u for fixed v. Now study the behavior of the function for $u \geq 0$ and its derivative.

(c) (Hölder's inequality/Schwarz inequality when $p = q = 2$) If $f, g \in \mathcal{R}(\alpha)$, then

$$\left| \int_a^b fg \, d\alpha \right| \leq \left\{ \int_a^b |f|^p \, d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q \, d\alpha \right\}^{1/q}.$$

Problem 76. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Show that $|f|$ is Riemann integrable and that

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx.$$

Problem 77. Let $f : [0, 1] \to \mathbb{R}$ be a continuously differentiable function. Prove that

$$\lim_{n \to \infty} \left(\sum_{k=1}^n f\left(\frac{k}{n}\right) \right) - n \int_0^1 f(x)dx = \frac{f(1) - f(0)}{2}.$$

Can we relax the condition of continuous differentiability?

Note: This tells us how fast this Riemann sums converge to the integral. In little o notation, we can say that

$$\sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x)dx + \frac{f(1) - f(0)}{2n} + o\left(\frac{1}{n}\right).$$

Problem 78. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose that

$$\int_a^b f(x)^2 dx = 0.$$

Show that $f(x) = 0$ for all $x \in [a, b]$.[32]

[32] This, together with the Cauchy-Schwarz inequality, shows that

$$\|f\|_2 = \left(\int_a^b f(x)^2 dx \right)^{1/2}$$

defines a norm on $C([a, b])$.

Problem 79. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Show that there is c in $[a, b]$ such that

$$\int_a^c f(x)dx = \int_c^b f(x)dx.$$

Note: This c is usually used in statistics to define the median of a random variable.

Problem 80. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that

$$\int_a^b f(x)dx = 0.$$

Show that there is $c \in (a, b)$ such that $f(c) = 0$.

Chapter 7: Sequences and Series of Functions

Problem 81. If $(f_n)_n$ is a sequence of unbounded functions, that is $\sup_{x \in E} |f_n(x)| = \infty$ for all n, can $(f_n)_n$ converge uniformly?

Problem 82 (Rudin 7.1). Prove that every uniformly convergent sequence of bounded functions $(f_n)_n$ is uniformly bounded, i.e., there is $C > 0$ such that

$$|f_n(x)| \le C, \quad \forall n \in \mathbb{N}, \forall x \in E.$$

Problem 83. Let $(f_n)_n$ be a sequence of uniformly continuous functions on E. Show that if $f \rightrightarrows f$ on E, then f is uniformly continuous on E.

Problem 84 (Rudin 7.7). For $n = 1, 2, 3, \ldots, x$ real, put

$$f_n(x) = \frac{x}{1 + nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n \to \infty} f_n'(x)$$

is correct if $x \ne 0$, but false if $x = 0$.

Problem 85 (Rudin 7.4 Modified). Let

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x^2}$$

In fact, $f(x) = \frac{\pi \coth \frac{\pi}{x} - 1}{2x}$. However, proving this claim requires extra work.

- Show that the series converge for any $x \ne 0$.

- Show that the series converge uniformly on $(-\infty, -c) \cup (c, \infty)$ for any $c > 0$.

- Is f bounded on $\mathbb{R} \setminus \{0\}$?

Problem 86. Let $g(x) = 1 - |x|$ for $x \in [-1, 1]$ and extend g to $\mathbb{R}$ by setting $g(x + 2) = g(x)$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by[33]

[33] This is called the Weierstrass function.

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n g(4^n x)$$

Prove that f is continuous everywhere but differentiable nowhere.
   Hint: Show that given

$$\frac{f(x + \delta_m) - f(x)}{\delta_m} \ge \frac{1}{2}\left(3^m + 1\right)$$

where $\delta_m = \frac{1}{2} 4^{-m}$ or $\delta_m = \frac{-1}{2} 4^{-m}$.

Problem 87. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. The goal for this problem is to show that there exists a sequence of polynomials $(p_n)_n$ such that $p_n \rightrightarrows f$ on $[0,1]$.[34]

- Define the Bernstein polynomials as

$$p_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

- Show that if $f(x) = 1$ or $f(x) = x$, then $p_n(x) = f(x)$ for all n.

- If $f(x) = x^2$, find a closed form for $p_n(x)$ and show that $p_n \rightrightarrows f$.

- Show that

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n} - x\right)^2 x^k (1-x)^{n-k} = \frac{x(1-x)}{n}.$$

- In the remainder of this problem, we will assume that f is a continuous function. Let $\varepsilon > 0$ and let $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$.

  – Write $p_n - f$ as a sum and split it into two parts: $k \in A_n$ and $k \in B_n$, where

  $$A_n = \left\{ k : \left|\frac{k}{n} - x\right| < \delta \right\}, \quad B_n = \left\{ k : \left|\frac{k}{n} - x\right| \geq \delta \right\}.$$

  – Bound the sum over A_n using continuity of f.

  – Bound the sum over B_n using the formula from the previous part.

  – Conclude that $p_n \rightrightarrows f$. [35]

Problem 88. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 x^n f(x)dx = 0, \quad \forall n \in \mathbb{N}.$$

Show that $f(x) = 0$ for all $x \in [0,1]$.
Hint: Use the previous problem.[36]

Problem 89. Let $f : [0, \pi] \to \mathbb{R}$ be a continuously differentiable function. Show that [37]

$$\lim_{n \to \infty} \int_0^{\pi} f(x) \sin(nx)dx = 0.$$

Problem 90. Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function. The goal of this problem is to extend the result in the previous problem to this case.

- Show that for any $\varepsilon > 0$, there are piecewise constant functions g, h such that

$$g \leq f \leq h, \quad \int_a^b h(x) - g(x)dx < \varepsilon.$$

[34] This is known as the Weierstrass approximation theorem. The textbook give an alternative proof using a convolution.

[35] This shows that the subspace of polynomials is dense in $C([0,1])$ with respect to the metric d_∞.

[36] This type of arguments is called a "Desnity argument", where we prove a statement on a dense subspace of X, then extend the result to X.

[37] This is a special case of the Riemann-Lebesgue lemma.

- Show that

$$\lim_{n\to\infty} \int_a^b g(x)\sin(nx)dx = \lim_{n\to\infty} \int_a^b h(x)\sin(nx)dx = 0.$$

- Conclude that $\int_a^b f(x)\sin(nx)dx \to 0$ as $n \to \infty$.

Problem 91. Let G be the following power series:[38]

$$G(x) = \sum_{n=0}^{\infty} a_n x^n,$$

[38] G is called the generating function of the sequence $(a_n)_n$.

Assume that the radius of convergence of G is 1 and that $\sum a_n$ converges. The goal of this problem is to show that G is continuous on $(-1,1]$. That is,[39]

$$\lim_{x\to 1^-} G(x) = G(1) = \sum_{n=0}^{\infty} a_n x^n.$$

[39] This is called Abel's theorem, we can also extended to the case $\sum a_n = \infty$. This is stated in the book in theorem 8.2.

- Show that we can assume without loss of generality that $\sum a_n = 0$. This will be assumed in the rest of the problem.

- Let $s_n = a_0 + \cdots + a_n$. Show that

$$G(x) = (1-x)\sum_{n=0}^{\infty} s_n x^n.$$

- Show that $G(x) \to 0$ as $x \to 1^-$. This will use the fact that $s_n \to 0$ and that $1 - x \to 0$ as $x \to 1^-$.

Problem 92. Using the previous show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \int_0^1 \frac{1}{1+x}dx, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \int_0^1 \frac{1}{1+x^2}dx.$$

The first one is called Mercator's series and it is equal to $\log 2$. The second one is called Gregory's series and it is equal to $\pi/4$.

Review

Problem 93. Let $(a_n)_n$ be the sequence defined by $a_1 = 1$ and $a_{n+1} = \sqrt{2 + a_n}$ for all $n \geq 1$. Show that the sequence converges and find its limit.

Problem 94. Let $(a_n)_n$ be a sequence such that for some $a_0 \in \mathbb{R}$ and $a_{n+1} = a_n - a_n^2$ for all $n \geq 0$. For which a_0 does the sequence converge?

Problem 95. Let
$$a_n = \frac{1}{n!} + \sum_{k=0}^{n} \frac{1}{k!}.$$

Show that $(a_n)_n$ is decreasing.

Problem 96. Let $x_1, x_2, \ldots, x_m$ be real positive numbers. Show that
$$\lim_{n \to \infty} \sqrt[n]{x_1^n + x_2^n + \cdots + x_m^n} = \max_{1 \leq i \leq m} x_i.$$

This justifies the definition of the infinity norm on $\mathbb{R}^m$ as
$$\|(x_1, x_2, \ldots, x_m)\|_\infty = \max_{1 \leq i \leq m} |x_i|.$$

Problem 97. Let
$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$

Write S_n as a Riemann sum, and find its limit.

Problem 98. Show that $n! \leq \left(\frac{n+1}{2}\right)^n$ for all $n \geq 2$, and deduce that
$$\frac{n! a^n}{(n+1)^n} \to 0,$$
for all $a \in [1, 2)$.

Problem 99. Let $f : X \to Y$ be a function that satisfies:

- For each compact $K \subset X$, $f(K)$ is compact.

- For every nested decreasing[40] sequence of compacts $(K_n)_n$ in X we have
$$f\left(\bigcap K_n\right) = \bigcap f(K_n).$$

[40] meaning $K_{n+1} \subset K_n$

Prove that f is continuous.
Note: Also think about whether the first condition (by itself) implies continuity.

Problem 100. Let $f : \mathbb{R} \to \mathbb{R}$ be a contraction[41] Show that the equation $f(x) = x$ has at most one solution.

[41] Meaning that there is $c < 1$ such that
$$|f(x) - f(y)| \leq c|x - y|$$
for all $x, y \in \mathbb{R}$.

Problem 101. Let $f : \mathbb{R} \to \mathbb{R}$ be a contraction. Show that the equation $f(x) = x$ has **exactly** one solution.

Problem 102. Let $f : [0,1] \to [0,1]$ be a continuous function. Such that $f(x)^2 = 1$ for all $x \in [0,1]$. Show that either $f(x) = 1$ for all x or $f(x) = -1$ for all x.

Problem 103. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at 0 and satisfies $f(2x) = f(x)$ for all $x \in \mathbb{R}$. Show that f is constant.
What if f was not continuous at 0?

Problem 104. Let $f : [a,b] \to \mathbb{R}$ be a continuous function, and let $\epsilon > 0$. Show that there is $k > 0$ such that for all $x, y \in [a,b]$ we have

$$|f(x) - f(y)| \leq k|x - y| + \epsilon.$$

Show that this may fail if $[a,b]$ is replaced by $\mathbb{R}$, or if $\epsilon = 0$.

Problem 105. Prove that $x^{-n}e^x \to 0$ as $x \to \infty$ for any $n \in \mathbb{N}$. [42]

[42] Use the series definition of e^x.

Problem 106. Prove that $x^n e^x \to 0$ as $x \to -\infty$ for any $n \in \mathbb{N}$.[43]

[43] Use the problem above.

Problem 107. Show that $x^n e^{-1/x^2} \to 0$ as $x \to 0$ for any $n \in \mathbb{N}$.

Problem 108. Let $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. Show that f is infinitely differentiable at 0 and that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.[44]

[44] Use the previous problem, and notice that $f^{(n)}(x)$ has a predictable form for $x \neq 0$.
  This shows that a function can be infinitely differentiable at a point, but its Taylor series at that point does not converge to the function in a neighborhood of that point.

Problem 109. Let $(x_n)_n$ be a sequence in a metric space X. Show that if $x_n \to \ell$ then every convergent subsequence of $(x_n)_n$ converges to x.

Problem 110. Let X be a compact metric space, and let $(x_n)_n$ be a sequence in X. Assume that every convergent subsequence of $(x_n)_n$ converge to the same point $\ell \in X$. Show that $x_n \to x$. [45]

[45] Compactness is important here. For example, $x_n = n((-1)^n - 1)$ does not converge.

Problem 111. For which values of a is the function below continuous at 0?

$$f(x) = \begin{cases} x^a \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Problem 112. Let $n \geq 2$ be an integer. Let $f : [0, \infty) \to \mathbb{R}$ be

$$f(x) = \frac{1 + x^n}{(1 + x)^n}.$$

(a) Find f' and study its sign.[46]

[46] You need to find when f' is positive and when it is negative.

(b) What is the minimum value of f?

(c) Prove that $(1 + x)^n \geq 2^{n-1}(1 + x^n)$ for all $x \geq 0$ and all $n \geq 2$.[47]

[47] The case $n = 1$ is trivial.

Problem 113. Let f be continuous on $[0,1]$ with values in $[a,b]$. Assume that $a < 0 < b$ and

$$\int_0^1 f(t)\, dt = 0.$$

Show that:

$$\int_0^1 f(t)^2\, dt \leq -ab.$$

Problem 114. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{n \to \infty} \int_0^1 x^n f(x) \, dx = 0.$$

Problem 115. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{n \to \infty} n \int_0^1 x^n f(x) \, dx = f(1).$$

Problem 116. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Find the following limit

$$\lim_{n \to \infty} \int_0^1 f(x^n) \, dx.$$

Problem 117. Give an example of a function $f : [a, b] \to \mathbb{R}$ that is differentiable but f' is not Riemann integrable.