Homework 7

Note: Always justify your answers.

Problem 1 (5 points). Let $(f_n)_n$ be a sequence of functions of bounded functions. If $f_n \rightrightarrows f$, show that f is bounded.

Solution 1. Since $f_n \rightrightarrows f$, for $\varepsilon = 1$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and for all x in the domain of the functions we have

$$|f_n(x) - f(x)| < 1,$$

In particular, for $n = N$ we have

$$|f_N(x) - f(x)| < 1 \implies |f(x)| < |f_N(x)| + 1.$$

Since f_N is bounded, there exists $M > 0$ such that for all x in the domain of the functions we have $|f_N(x)| \leq M$. Therefore, for all x in the domain of the functions we have

$$|f(x)| < M + 1,$$

which shows that f is bounded.

Problem 2 (15 points)**.** This is a multipart problem to show the need for a bounded interval in theorem 7.17.

(a) Find a differentiable and increasing function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = 0$ for $x < 0$ and $g(x) = 1$ for $x > 1$. Hint: You can choose g to be a cubic polynomial on $[0, 1]$.

(b) Use the previous part to construct a sequence of functions $(f_n)_n$ such that f_n is increasing and differentiable for all n, $f_n(x) = 0$ for $x < n$ and $f_n(x) = 1$ for $x > 2n$.

(c) Show that f_n' converges uniformly to 0, there is x_0 such that $f_n(x_0)$ converges, but f_n does not converge uniformly.

Solution 2. (a) We want $g'(0) = g'(1) = g(0) = 0$ and $g(1) = 1$. A cubic polynomial has the form $g(x) = ax^3 + bx^2 + cx + d$. The conditions give us the following system of equations:

$$\begin{aligned} d &= 0 \\ c &= 0 \\ a + b + c + d &= 1 \\ 3a + 2b + c &= 0 \end{aligned}$$

Solving this system, we get $a = -2$, $b = 3$, $c = 0$ and $d = 0$. Therefore, we can define

$$g(x) = \begin{cases} 0 & x < 0 \\ 3x^2 - 2x^3 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}.$$

It is easy to check that g is differentiable and increasing.

Note: The existence of a cubic polynomial is actually guaranteed for any choice of $g(0), g(1), g'(0), g'(1)$. A such polynomial is called a Hermite interpolating polynomial, and it is usually used to construct splines (smooth curves passing through a set of points).

(b) We can define

$$f_n(x) = g\left(\frac{x - n}{n}\right).$$

It is easy to check that f_n is increasing and differentiable for all n, $f_n(x) = 0$ for $x < n$ and $f_n(x) = 1$ for $x > 2n$.

(c) We have

$$f_n'(x) = \frac{1}{n} g'\left(\frac{x - n}{n}\right).$$

Since g' is bounded, we have that f_n' converges uniformly to 0.

For $x_0 = 0$, we have that $f_n(0) = 0$ for all n, so $f_n(0)$ converges to 0. However, for any fixed x, if we choose $n > x$, then we have $f_n(x) = 1$. Therefore, for any fixed x, the sequence $(f_n(x))_n$ does not converge to 0. This shows that f_n does not converge uniformly.

Note: This shows that 7.17 requires a bounded interval.

Problem 3 (20 points)**.** Let $(p_n)_n$ be a uniformly convergent sequence of polynomials on $\mathbb{R}$. Prove that the limit function is a polynomial.

Hint: Try to use the unboundedness of the domain to your advantage.

Hint: Use Cauchy's criterion for uniform convergence.

Solution 3. The most important properties of polynomials that we will use are that the difference of two polynomials is a polynomial, and that any non-constant polynomial is unbounded on $\mathbb{R}$.

Let $\varepsilon = 1$. Since $(p_n)_n$ converges uniformly, by Cauchy's criterion there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$ and for all $x \in \mathbb{R}$ we have

$$|p_n(x) - p_m(x)| < 1.$$

But $p_n - p_m$ is a polynomial, so if it is not constant, then it is unbounded on $\mathbb{R}$. Therefore, for the inequality above to hold, $p_n - p_m$ must be constant. In conclusion, all polynomials p_n with $n \geq N$ differ from each other by a constant.

Let $c_n = p_n - p_N$ for $n \geq N$. Then by the above observation, c_n is real number for all $n \geq N$. Since p_n converges to some function p, we have

$$p(x) - p_N(x) = \lim_{n\to\infty} p_n(x) - P_N(x) = \lim_{n\to\infty} c_n.$$

This shows that $(c_n)_n$ converges to some real number ℓ. Moreover, it shows that the $p(x) = p_N(x) + \ell$ for all $x \in \mathbb{R}$. Therefore, p is a polynomial.

Note: This basically says that the degree of p_n is eventually constant.

Note: This shows that the Weierstrass approximation theorem cannot be extended to the whole real line (or even to $[a, \infty)$). To see this, just consider a non-polynomial function $f : \mathbb{R} \to \mathbb{R}$, then there is no sequence of polynomials converging uniformly to f.

Problem 4 (20 points)**.** Let $(f_n)_n$ be the sequence of functions defined by

$$f_n(x) = \left(1 + \frac{x}{n}\right)^n.$$

Show that f_n converge pointwise for every $x \in \mathbb{R}$, but the convergence is not uniform on $\mathbb{R}$.

Hint: For the first part, you can fix x and show that $(f_n(x))_n$ is monotone and bounded. For the second part, you should ask yourself what kind of function is $f_n(x)$.

Solution 4. We first need the following version of Bernoulli's inequality: $(1+x)^n \geq 1 + nx$ for all $x \geq -1$ and all $n \in \mathbb{N}$. This can be proven in at least two ways:

- By induction on n. It is true for $n = 1$. Assume it is true for some n. Then

  $$(1+x)^{n+1} = (1+x)^n(1+x) \geq (1+nx)(1+x) = 1 + (n+1)x + nx^2 \geq 1 + (n+1)x.$$

  Here, we have used $1 + x \geq 0$.

- Using calculus: Define $g(x) = (1+x)^n - 1 - nx$. Then $g(-1) = -1 + n \geq 0$ and

  $$g'(x) = n((1+x)^{n-1} - 1).$$

  So, g is decreasing on $[-1, 0]$ and increasing on $[0, \infty)$. Thus, $g(x) \geq g(0) \geq 0$ for all $x \geq -1$.

Now, let go back to the problem at hand. Let $x \in \mathbb{R}$ be fixed. Let n large enough such that $f_n(x) \neq 0$ (this is only an issue if $x = -n$ for some n, but we can just choose $n > |x|$). We have

$$\frac{f_{n+1}(x)}{f_n(x)} = \left(1 + \frac{x}{n+1}\right)^{n+1} \left(1 + \frac{x}{n}\right)^{-n} = \left(\frac{n(n+1+x)}{(n+1)(n+x)}\right)^n \left(1 + \frac{x}{n+1}\right) = \left(1 - \frac{x}{(n+1)(n+x)}\right)^n \left(1 + \frac{x}{n+1}\right).$$

Then by Bernoulli's inequality, we have

$$\frac{f_{n+1}(x)}{f_n(x)} \geq \left(1 - \frac{nx}{(n+1)(n+x)}\right)\left(1 + \frac{x}{n+1}\right) = 1 + \frac{x^2}{(n+1)(n+x)} \geq 1.$$

Thus, the sequence $(f_n(x))_n$ is increasing. Moreover, by the binomial theorem we have

$$f_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{n^k} = \sum_{k=0}^{n} \frac{x^k}{k!} \cdot \frac{(n)(n-1)\cdots(n-k+1)}{n^k} \leq \sum_{k=0}^{n} \frac{x^k}{k!} \leq \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

The sum on the right converges, so $(f_n(x))_n$ is bounded. Therefore, $(f_n(x))_n$ converges for every fixed $x \in \mathbb{R}$.

Note that $f_n(x)$ is a polynomial of degree n in x. If the convergence was uniform on $\mathbb{R}$, then by problem 3 the degree of the polynomial f_n must be eventually constant. This is not the case, so the convergence cannot be uniform on $\mathbb{R}$.

Problem 5 (20 points)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(a) Show that the series converge uniformly on $[-c, c]$.

(b) Show that f is differentiable and that $f' = f$.

(c) Show that

$$f(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n.$$

Hint: Use the binomial theorem.

(d) Show that the sequence in the previous problem converges uniformly to f on every bounded interval.

Hint: Use Dini's theorem (theorem 7.13 in the book).

Solution 5.

(a) For $x \in [-c, c]$, we have $\left|\frac{x^n}{n!}\right| \leq \frac{c^n}{n!}$. Since the series $\sum \frac{c^n}{n!}$ converges, by the Weierstrass M-test the series $\sum \frac{x^n}{n!}$ converges uniformly on $[-c, c]$.

(b) This was discussed in class: The series of derivatives converges uniformly on $[-c, c]$ for any $c > 0$, so we can differentiate the series term by term. We have

$$f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x).$$

(c) Using the binomial theorem, we have

$$\sum_{k=0}^{n} \frac{x^k}{k!} - \left(1 + \frac{x}{n}\right)^n = \sum_{k=0}^{n} \frac{x^k}{k!} \left(1 - \frac{(n)(n-1)\cdots(n-k+1)}{n^k}\right) = \sum_{k=0}^{n} \frac{x^k}{k!} \underbrace{\left(1 - \prod_{j=0}^{k-1} \left(1 - \frac{j}{n}\right)\right)}_{\text{call it } b_{k,n}}. \quad (1)$$

Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\sum_{k=N+1}^{n} \frac{|x|^k}{k!} < \varepsilon.$$

Then, because $b_{k,n} \in [0, 1]$ for all k, n, we have

$$\sum_{k=N}^{n} \left|\frac{x^k}{k!} b_{k,n}\right| \leq \varepsilon$$

On the other hand, we have for any $k \leq N$

$$b_{k,n} = 1 - \prod_{j=0}^{k-1} \left(1 - \frac{j}{n}\right) \leq 1 - \left(1 - \frac{N-1}{n}\right)^k \leq 1 - \left(1 - \frac{N-1}{n}\right)^N.$$

Now, let us go back to (1). For $n \geq N$, we have

$$\left|\sum_{k=0}^{n} \frac{x^k}{k!} - \left(1 + \frac{x}{n}\right)^n\right| \leq \varepsilon + \left(1 - \left(1 - \frac{N-1}{n}\right)^N\right) \sum_{k=0}^{N} \frac{|x|^k}{k!}.$$

The right tends to ε as $n \to \infty$. Since ε is arbitrary, we have shown that

$$f(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n.$$

(d) Let $g_n(x) = \left(1 + \frac{x}{n}\right)^n$. On any compact interval $[-c, c]$, g_n converges pointwise to f, which is continuous (By the Weierstrass M-test). Moreover, g_n is an (eventually) increasing sequence of functions Therefore, $(g_n)_n$ converges uniformly to f on $[-c, c]$ by Dini's theorem.

Problem 6 (20 points)**.** Let $f(x) = \sqrt{x}$ on $[0,1]$. Let $(p_n)_n$ be the sequence of polynomials on $[0,1]$ defined by the recurrence relation

$$p_0(x) = 0, \quad p_{n+1}(x) = p_n(x) + \frac{x - p_n(x)^2}{2}.$$

Show that p_n converges uniformly to f on $[0,1]$.

Hint: You can follow the same steps we have done in class for $f(x) = |x|$.

Solution 6. We have

$$\sqrt{x} - p_{n+1}(x) = \sqrt{x} - p_n(x) - \frac{x - p_n(x)^2}{2} = (\sqrt{x} - p_n(x))\left(1 - \frac{\sqrt{x} + p_n(x)}{2}\right). \tag{2}$$

Therefore, if $p_n \leq f$, then $p_{n+1} \leq f$. Since $p_0 \leq f$, by induction we have $p_n \leq f$ for all n. This also shows that $(p_n)_n$ is increasing. Meaning $p_{n+1}(x) \geq p_n(x)$ for all $x \in [0,1]$ and all n. Which implies that all polynomials p_n are non-negative on $[0,1]$.

Using the non-negativity of p_n and (2), we have

$$0 \leq \sqrt{x} - p_{n+1}(x) \leq (\sqrt{x} - p_n(x))\left(1 - \frac{\sqrt{x}}{2}\right).$$

Then

$$0 \leq \sqrt{x} - p_n(x) \leq \sqrt{x}\left(1 - \frac{\sqrt{x}}{2}\right)^n.$$

Similar to the case of $f(x) = |x|$, we consider $g(t) = t(1 - \frac{t}{2})^n$ for $t \in [0,1]$. We have

$$g'(t) = \frac{((n+1)t - 2)}{t-2}\left(1 - \frac{t}{2}\right)^{n-1}.$$

Which shows that g has a maximum at $t = \frac{2}{n+1}$. Therefore,

$$\max_{x \in [0,1]}(\sqrt{x} - p_n(x)) \leq g\left(\frac{2}{n+1}\right) = \frac{2}{n+1}\left(1 - \frac{1}{n+1}\right)^n \leq \frac{2}{n+1}.$$

Therefore, $p_n \rightrightarrows f$ on $[0,1]$.

Note: We could use Dini's theorem to conclude the uniform convergence, since $(p_n)_n$ is an increasing sequence of continuous functions converging pointwise to a continuous function f on the compact interval $[0,1]$.