

HOMEWORK 7

Note: Always justify your answers.

Problem 1 (5 points). Let $(f_n)_n$ be a sequence of functions of bounded functions. If $f_n \rightrightarrows f$, show that f is bounded.

Problem 2 (15 points). This is a multipart problem to show the need for a bounded interval in theorem 7.17.

- (a) Find a differentiable and increasing function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g(x) = 0$ for $x < 0$ and $g(x) = 1$ for $x > 1$. Hint: You can choose g to be a cubic polynomial on $[0, 1]$.
- (b) Use the previous part to construct a sequence of functions $(f_n)_n$ such that f_n is increasing and differentiable for all n , $f_n(x) = 0$ for $x < n$ and $f_n(x) = 1$ for $x > 2n$.
- (c) Show that f'_n converges uniformly to 0, there is x_0 such that $f_n(x_0)$ converges, but f_n does not converge uniformly.

Problem 3 (20 points). Let $(p_n)_n$ be a uniformly convergent sequence of polynomials on \mathbb{R} . Prove that the limit function is a polynomial.

Hint: Try to use the unboundedness of the domain to your advantage.

Hint: Use Cauchy's criterion for uniform convergence.

Problem 4 (20 points). Let $(f_n)_n$ be the sequence of functions defined by

$$f_n(x) = \left(1 + \frac{x}{n}\right)^n.$$

Show that f_n converge pointwise for every $x \in \mathbb{R}$, but the convergence is not uniform on \mathbb{R} .

Hint: For the first part, you can fix x and show that $(f_n(x))_n$ is monotone (for large enough n) and bounded. For the second part, you should ask yourself what kind of function is $f_n(x)$.

Problem 5 (20 points). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

- (a) Show that the series converge uniformly on $[-c, c]$.
- (b) Show that f is differentiable and that $f' = f$.
- (c) Show that

$$f(x) = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n.$$

Hint: Use the binomial theorem.

- (d) Show that the sequence in the previous problem converges uniformly to f on every bounded interval.

Hint: Use Dini's theorem (theorem 7.13 in the book).

Problem 6 (20 points). Let $f(x) = \sqrt{x}$ on $[0, 1]$. Let $(p_n)_n$ be the sequence of polynomials on $[0, 1]$ defined by the recurrence relation

$$p_0(x) = 0, \quad p_{n+1}(x) = p_n(x) + \frac{x - p_n(x)^2}{2}.$$

Show that p_n converges uniformly to f on $[0, 1]$.

Hint: You can follow the same steps we have done in class for $f(x) = |x|$.