Homework 6

Note: Always justify your answers.

Problem 1 (30pts)**.** Let $f : [0,1] \to \mathbb{R}$ be the Thomae function defined by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ in lowest terms} \\ 0 & \text{if } x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$$

Here, "the lowest terms" means that p and q are co-prime integers. The goal of this problem is to show that f is Riemann integrable on $[0,1]$ and $\int_0^1 f(x)dx = 0$.

(a) Let $\varepsilon > 0$. Show that there are finitely many numbers $x \in [0,1]$ such that $f(x) \geq \varepsilon$.

(b) Based on the previous part, find a partition P of $[0,1]$ such that $U(f,P) < \varepsilon$ (or some multiple of ε).

(c) Show that the upper integral of f is zero, and conclude that f is Riemann integrable on $[0,1]$ and $\int_0^1 f(x)dx = 0$.

Solution 1.

(a) Let ε, then $f(x) \geq \varepsilon$ is equivalent to $x = \frac{p}{q}$ with $q \leq \varepsilon^{-1}$ (only finitely many q's). Since $x \in [0,1]$, we have $p \leq q\varepsilon^{-1}$. Together with the previous statement shows that there are finitely many x's such that $f(x) \geq \varepsilon$.

(b) Let $\{z_1, z_2, \ldots, z_k\} = \{x | f(x) \geq \varepsilon\}$. Let P be a partition where $\Delta x_i = \frac{\varepsilon}{k}$, then the sum defining $U(P,f)$ has at most k non-zero terms and $f(x) \leq 1, \forall x \in [0,1]$, which means that $U(P,f) \leq k \cdot 1 \cdot \frac{\varepsilon}{k} = \varepsilon$.

(c) Let $\varepsilon > 0$ and let P the partition defined in the previous part. Then $0 \leq L(P,f) \leq U(P,f) \leq \varepsilon$. Then, $U(P,f) - L(P,f) \leq \varepsilon$. Which leads directly to the integrability of f by theorem 6.6.

Problem 2 (20pts). Let $d : \mathfrak{R} \times \mathfrak{R} \to \mathbb{R}$ be defined by

$$d(f,g) = \sqrt{\int_a^b (f-g)^2 dx}$$

Prove that

$$d(f,g) \leq d(f,h) + d(h,g), \qquad \text{for all } f,g,h \in \mathfrak{R}.$$

Hint: You can use Holder's inequality (see recitation).

Note: (not part of the problem) This means that d satisfies the triangle inequality. Is d a metric on $\mathfrak{R}$? Is there a subset of $\mathfrak{R}$ on which d is a metric? Check problem 8 in the recitation sheet for a hint.

Solution 2. This is equivalent to showing that

$$\sqrt{\int_a^b (u+v)^2 dx} \leq \sqrt{\int_a^b u^2 dx} + \sqrt{\int_a^b v^2 dx}$$

(Here, $u = f - h$ and $v = h - g$). Start from Cauchy-Schwarz's inequality

$$\int_a^b uv dx \leq \sqrt{\int_a^b u^2 dx} \cdot \sqrt{\int_a^b v^2 dx}$$

Multiply both sides by 2 and add $\int u^2 + v^2$ to both sides

$$\int_a^b u^2 + v^2 + 2uv dx \leq \int_a^b u^2 dx + \int_a^b v^2 dx + 2\sqrt{\int_a^b u^2 dx} \cdot \sqrt{\int_a^b v^2 dx},$$

which is equivalent to

$$\int_a^b (u+v)^2 dx \leq \left(\sqrt{\int_a^b u^2 dx} + \sqrt{\int_a^b v^2 dx} \right)^2.$$

Problem 3 (20pts)**.** Let $f, g : [-a, a] \to \mathbb{R}$ be Riemann integrable functions. Assume that f is an even function and g is an odd function. Show that

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx, \qquad \int_{-a}^{a} g(x)dx = 0.$$

Solution 3. This is a direct consquence of the next problem and a u-substitution. For instance, if f is even, then $f(-x) = f(x)$, which means that

$$\int_{-a}^{0} f(x)dx = \int_{0}^{a} f(-u)du = \int_{0}^{a} f(u)du.$$

Then,

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

If g is odd, then $g(-x) = -g(x)$, and

$$\int_{-a}^{0} g(x)dx = \int_{0}^{a} g(-u)du = -\int_{0}^{a} g(u)du.$$

This means that $\int_{-a}^{a} g(x)dx = \int_{0}^{a} g(x)dx - \int_{0}^{a} g(x)dx = 0$.

Problem 4 (30pts). Let $f \in \mathfrak{R}(\alpha)$ on $[a,b]$ and let $c \in (a,b)$.

(a) Show that $f \in \mathfrak{R}(\alpha)$ on $[a,c]$ and $[c,b]$.

(b) Show that $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$.

Hint: Always start with $\varepsilon > 0$. In the first part, start with a partition P of $[a,b]$ such that the upper sum and lower sum are close. Then, add c to the partition. You can do something similar for the second part.

Solution 4.

(a) Let $\varepsilon > 0$ and let P be a partition of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$.

Let $P^* = P \cup \{c\}$, then P^* is a partition of $[a,b]$ and $U(P^*,f,\alpha) - L(P^*,f,\alpha) < \varepsilon$ (this follows from theorem 6.7.a).

Let P_1 be the partition of $[a,c]$ formed by the points in P^* that are in $[a,c]$. Then

$$U(P_1,f,\alpha) - L(P_1,f,\alpha) \leq U(P^*,f,\alpha) - L(P^*,f,\alpha) < \varepsilon.$$

This is because the left hand side is a sum of some of the terms in the right hand side (see the definition of upper and lower sums). This shows that $f \in \mathfrak{R}(\alpha)$ on $[a,c]$. Following a similar idea, we can construct a partition P_2 of $[c,b]$ and show that $f \in \mathfrak{R}(\alpha)$ on $[c,b]$.

(b) Following the previous part, we fix $\varepsilon > 0$ and let P^* be a partition containing c such that $U(P^*,f,\alpha) - L(P^*,f,\alpha) < \varepsilon$. We break it down into two partitions P_1 and P_2 of $[a,c]$ and $[c,b]$ respectively. Then,

$$\left| \int_a^b f d\alpha - \int_a^c f d\alpha - \int_c^b f d\alpha \right| \leq \varepsilon.$$

Since this holds for all $\varepsilon > 0$, we obtain the desired result.