Homework 6

Note: Always justify your answers.

Problem 1 (30pts). Let $f : [0,1] \to \mathbb{R}$ be the Thomae function defined by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ in lowest terms} \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

Here, "the lowest terms" means that p and q are co-prime integers. The goal of this problem is to show that f is Riemann integrable on $[0,1]$ and $\int_0^1 f(x)dx = 0$.

(a) Let $\varepsilon > 0$. Show that there are finitely many numbers $x \in [0,1]$ such that $f(x) \geq \varepsilon$.

(b) Based on the previous part, find a partition P of $[0,1]$ such that $U(f,P) < \varepsilon$ (or some multiple of ε).

(c) Show that the upper integral of f is zero, and conclude that f is Riemann integrable on $[0,1]$ and $\int_0^1 f(x)dx = 0$.

Problem 2 (20pts). Let $d : \mathfrak{R} \times \mathfrak{R} \to \mathbb{R}$ be defined by

$$d(f,g) = \sqrt{\int_a^b (f-g)^2 dx}$$

Prove that

$$d(f,g) \leq d(f,h) + d(h,g), \qquad \text{for all } f,g,h \in \mathfrak{R}.$$

Hint: You can use Holder's inequality (see recitation).

Note: (not part of the problem) This means that d satisfies the triangle inequality. Is d a metric on $\mathfrak{R}$? Is there a subset of $\mathfrak{R}$ on which d is a metric? Check problem 8 in the recitation sheet for a hint.

Problem 3 (20pts). Let $f, g : [-a,a] \to \mathbb{R}$ be Riemann integrable functions. Assume that f is an even function and g is an odd function. Show that

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx, \qquad \int_{-a}^{a} g(x)dx = 0.$$

Problem 4 (30pts). Let $f \in \mathfrak{R}(\alpha)$ on $[a,b]$ and let $c \in (a,b)$.

(a) Show that $f \in \mathfrak{R}(\alpha)$ on $[a,c]$ and $[c,b]$.

(b) Show that $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$.

Hint: Always start with $\varepsilon > 0$. In the first part, start with a partition P of $[a,b]$ such that the upper sum and lower sum are close. Then, add c to the partition. You can do something similar for the second part.