Homework 5

Note: Always justify your answers.

Problem 1 (20pts). Let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$|f(x) - f(y)| \le (x-y)^2, \qquad \forall x, y \in \mathbb{R}.$$

Prove that f is constant.

Solution 1. It follows from the inequality that

$$\left|\frac{f(x) - f(y)}{x-y}\right| \le |x-y|, \qquad \forall x, y \in \mathbb{R} \text{ such that } x \ne y$$

which implies that f is differentiable everywhere and $f'(x) = 0$ for all $x \in \mathbb{R}$. By MVT, given any $x, y \in \mathbb{R}$,

$$f(x) - f(y) = 0(x-y) = 0,$$

which is the same as saying that f is constant.

Problem 2 (20pts). Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function defined on an interval I. Prove that if f' is bounded on I, then f is uniformly continuous on I.

Hint: : Prove that f is Lipschitz continuous on I (see the previous homework for a definition of Lipschitz continuity).

Solution 2. Let $M = \sup |f'|$. By MVT, given any $x, y \in I$, there is c between them such that

$$|f(x) - f(y)| = |f'(c)||x - y|.$$

Therefore,

$$|f(x) - f(y)| \leq M|x - y|,$$

which means that f is Lipschitz continuous (Hence, uniformly continuous).

Note: The boundedness of the derivative is crucial here. For instance, $x \mapsto \sqrt{x}$ is differentiable on $(0, 1]$ but not Lipschitz continuous (This does not contradict this problem since the derivative is unbounded).

Problem 3 (20pts). Suppose f is defined in a neighborhood of x and that $f''(x)$ exists. Show that

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x).$$

After that, show by an example that the limit may exist even if $f''(x)$ does not.

Solution 3. Fix x and $\varepsilon > 0$. Let $g(h) = f(x+h) + f(x-h) - 2f(x)$. Then, g is defined on a neighborhood I of zero, and $g(0) = g'(0) = 0$. The problem is asking us to show that $\frac{g(h)}{h^2} \to f''(x)$ as $h \to 0$.

By taking the derivative twice, we have $g''(0) = 2f''(x)$. Therefore, there is $\delta > 0$ such that

$$\left|\frac{g'(h)}{h} - 2f''(x)\right| \leq \varepsilon, \qquad \forall h \in (-\delta, \delta) \subset I.$$

Now, let $h \in (-\delta, \delta)$ and consider the function $r(z) = g(z)h^2 - g(h)z^2$ defined on $(-\delta, \delta)$. Then, by construction we have $r(0) = r(h) = 0$, so by Rolle's theorem, there is c between h and 0 (Hence, in $(-\delta, \delta)$) such that $r'(c) = g'(c)h^2 - 2cg(h) = 0$, which can be rearranged as

$$\frac{g(h)}{h^2} = \frac{g'(c)}{2c}.$$

Now, we have

$$\left|\frac{g(h)}{h^2} - f''(x)\right| = \left|\frac{g'(c)}{2c} - f''(x)\right| \leq \frac{\varepsilon}{2},$$

which shows that $\frac{g(h)}{h^2} \to f''(x)$ as $h \to 0$.

Note: The introduction of the auxiliary function r is the idea behind the so-called L'Hopital's rule.

Problem 4 (20pts)**.** Let $f(x) = x^n$ and $g(x) = x^{1/n}$ be defined on their domain ($\mathbb{R}$ for f, and $[0, \infty)$ for g) where $n \in \mathbb{N}$.

(a) Use the definition of derivative to prove that $f'(x) = nx^{n-1}$ for all $x \in \mathbb{R}$.

(b) Use the definition of derivative to prove that $g'(x) = \frac{1}{n}x^{\frac{1}{n}-1}$ for all $x > 0$, and prove that g is not differentiable at $x = 0$ when $n > 1$.

(c) Combine the above two results to prove the function $h(x) = x^r$, where $r > 0$ is a rational number, is differentiable on $(0, \infty)$ and $h'(x) = rx^{r-1}$ for all $x > 0$.

Solution 4. Parts (a) and (b) rely on the following identity

$$x^n - y^n = (x - y)\left(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\right). \tag{1}$$

(a) Let $x \in \mathbb{R}$. By definition and using (1), we have

$$f'(x) = \lim_{y \to x} \frac{x^n - y^n}{x - y} = \lim_{y \to x} \sum_{k=0}^{n-1} x^{n-1-k}y^k = nx^{n-1}.$$

Here, we have used the fact that $x \mapsto x^k$ is continuous for any $k \in \mathbb{N}$.

(b) Notice that $x^{\frac{1}{n}} - y^{\frac{1}{n}}$ can be factored as (this follows from (1) by replacing x and y by $x^{\frac{1}{n}}$ and $y^{\frac{1}{n}}$, respectively)

$$x^{\frac{1}{n}} - y^{\frac{1}{n}} = (x - y)\frac{1}{x^{\frac{n-1}{n}} + x^{\frac{n-2}{n}}y^{\frac{1}{n}} + \cdots + y^{\frac{n-1}{n}}}.$$

Therefore, for any $x > 0$, we have

$$\lim_{y \to x} \frac{g(x) - g(y)}{x - y} = \frac{1}{nx^{\frac{n-1}{n}}} = \frac{1}{n}x^{\frac{1}{n}-1}.$$

(c) Let $r = m/n$ with $p, q \in \mathbb{N}$. The claim follows from chain rule since $h = g \circ f$, where $g : x \mapsto x^{1/n}$ and $f : x \mapsto x^m$. Therefore,

$$h'(x) = g'(f(x))f'(x) = \frac{1}{n}(x^m)^{\frac{1}{n}-1} \cdot mx^{m-1} = \frac{m}{n}x^{\frac{m}{n}-1} = rx^{r-1}.$$

Problem 5 (10pts)**.** Let p be a differentiable function from $\mathbb{R}$ to $\mathbb{R}$. Show that if p has n distinct real zeros, then its derivative p' has at least $n-1$ distinct real.

Note: We say that $a \in \mathbb{R}$ is a zero of p if $p(a) = 0$.

Solution 5. Let $x_1 < x_2 < \cdots < x_n$ be zeros of p. Let $2 \leq i \leq n$, then by Rolle's theorem, there is $c_i \in (x_{i-1}, x_i)$ such that $p'(c_i) = 0$. These c_i's are distinct since the intervals (x_{i-1}, x_i) are disjoint. Hence, p' has (at least) $n-1$ zeros (which are $c_2, c_3, \ldots, c_n$).

Problem 6 (10pts). Let $f : [0, 2] \to \mathbb{R}$ be continuous on $[0, 2]$ and differentiable twice on $(0, 2)$. Show that if $f(0) = 0, f(1) = 1$ and $f(2) = 2$, then there exists some $c \in (0, 2)$ such that $f''(c) = 0$.

Note: "Differentiable twice" means that f is differentiable, and the first derivative f' is differentiable.

Solution 6. By MVT, there is $c_1 \in (0, 1)$ such that $f'(c_1) = 1$, and there is $c_2 \in (1, 2)$ such that $f'(c_2) = 1$. By Rolle's theorem (or MVT), there is $c_3 \in (c_1, c_2)$ such that $f''(c_3) = 0$.