# Homework 5

**Note:** Always justify your answers.

**Problem 1** (20pts). Let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$|f(x) - f(y)| \leq (x - y)^2, \qquad \forall x, y \in \mathbb{R}.$$

Prove that $f$ is constant.

**Problem 2** (20pts). Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function defined on an interval $I$. Prove that if $f'$ is bounded on $I$, then $f$ is uniformly continuous on $I$.

*Hint:* : Prove that $f$ is Lipschitz continuous on $I$ (see the previous homework for a definition of Lipschitz continuity).

**Problem 3** (20pts). Suppose $f$ is defined in a neighborhood of $x$ and that $f''(x)$ exists. Show that

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

After that, show by an example that the limit may exist even if $f''(x)$ does not.

**Problem 4** (20pts). Let $f(x) = x^n$ and $g(x) = x^{1/n}$ be defined on their domain ($\mathbb{R}$ for $f$, and $[0, \infty)$ for $g$) where $n \in \mathbb{N}$.

(a) Use the definition of derivative to prove that $f'(x) = nx^{n-1}$ for all $x \in \mathbb{R}$.

(b) Use the definition of derivative to prove that $g'(x) = \frac{1}{n}x^{\frac{1}{n}-1}$ for all $x > 0$, and prove that $g$ is not differentiable at $x = 0$ when $n > 1$.

(c) Combine the above two results to prove the function $h(x) = x^r$, where $r > 0$ is a rational number, is differentiable on $(0, \infty)$ and $h'(x) = rx^{r-1}$ for all $x > 0$.

**Problem 5** (10pts). Let $p$ be a differentiable function from $\mathbb{R}$ to $\mathbb{R}$. Show that if $p$ has $n$ distinct real zeros, then its derivative $p'$ has at least $n - 1$ distinct real.

*Note:* We say that $a \in \mathbb{R}$ is a zero of $p$ if $p(a) = 0$.

**Problem 6** (10pts). Let $f : [0, 2] \to \mathbb{R}$ be continuous on $[0, 2]$ and differentiable twice on $(0, 2)$. Show that if $f(0) = 0, f(1) = 1$ and $f(2) = 2$, then there exists some $c \in (0, 2)$ such that $f''(c) = 0$.

*Note:* "Differential twice" means that $f$ is differentiable, and the first derivative $f'$ is differentiable.