Homework 4

Note: Always justify your answers.

Problem 1 (20pts)**.** Find the domain of convergence for the following power series:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty} nx^n$

(c) $\displaystyle\sum_{n=0}^{\infty} n!x^n$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt[n]{n}}$

Solution 1.

(a) The radius is 1 by the ratio test: $\frac{n+1}{n} \to 1$. You can also use the root test here. The power series converge for $x = -1$ by the alternating series test, and diverge for $x = 1$ (Harmonic series). Hence, the domain is $[-1, 1)$.

(b) The radius is 1 by the ratio test, but the power series diverge for $x = \pm 1$ (using the divergence test). Hence, the domain of convergence is $(-1, 1)$.

(c) The limit of $\frac{c_{n+1}}{c_n}$ is ∞. Therefore, the radius of convergence is zero and the domain of convergence is $\{0\}$.

(d) Recall that $\sqrt[n]{n} \to 1$. Therefore, $\left|\frac{c_{n+1}}{c_n}\right| \to 1$, and then the radius of convergence is 1. The series does not coverge for $x = \pm 1$ by the divergence test, so the domain of convergence is $(-1, 1)$.

Problem 2 (20 pts)**.** Let $(a_n)_n$ and $(b_n)_n$ be two positive sequences such that

$$\lim_n \frac{a_n}{b_n} = \ell \in \mathbb{R}\setminus\{0\}.$$

Show that the series $\sum_n a_n$ converges if and only if the series $\sum_n b_n$ converges.

Solution 2. First, we note that $\ell > 0$ (Basically, $\leq \geq 0$ since $[0, \infty)$ is closed, and we know that $\ell \neq 0$). Let $\varepsilon = \frac{\ell}{2} > 0$. Then, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left|\frac{a_n}{b_n} - \ell\right| < \varepsilon \implies \frac{\ell}{2} \leq \frac{a_n}{b_n} \leq \frac{3\ell}{2}.$$

So, for all $n \geq N$, we have

$$C_1 a_n \leq b_n \leq C_2 a_n, \qquad C_1, C_2 > 0.$$

if $\sum a_n$ converges, then by comparison test $\sum b_n$ converges since $b_n \leq C_2 a_n$ for all $n \geq N$. Similarly, if $\sum b_n$ converges, then by comparison test $\sum a_n$ converges since $a_n \leq \frac{1}{C_1} b_n$ for all $n \geq N$.

Problem 3 (20 pts)**.** Let $f : [0,1] \to [0,1]$ be a continuous function. Show that the equation $f(x) = x$ has at least one solution.

Solution 3. Let $g(x) = x - f(x)$, then $g(0) = 1 - f(1) \geq 0$ and $g(1) = -f(0) \leq 0$. Since g is continuous (the difference of two continuous functions is continuous). Then, there is $c \in [0,1]$ such that $g(c) = 0$.

Problem 4 (20 pts)**.** Let $f : X \to Y$ be a function from a metric space (X, d_X) to a metric space (Y, d_Y) (meaning that d_X and d_Y are the metrics on X and Y, respectively). We say that f is *Lipschitz continuous* if there exists a constant $L > 0$ such that for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leq L d_X(x_1, x_2).$$

Show that if f is Lipschitz continuous, then it is continuous (you can also show that it is uniformly continuous if you want to).

Solution 4. Given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{L}$. Then, if $x, y \in X$ with the property that $d_X(x, y) < \delta$, we have $d_Y(f(x), f(y)) < \varepsilon$.

Problem 5 (20 pts)**.** Is $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = \sqrt{x}$ uniformly continuous? Justify your answer.

Same question for $f(x) = x^2$.

Solution 5. Recall that $\left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x-y|}$. Then, for any $\varepsilon > 0$, we can choose $\delta = \varepsilon^2$ to ensure that

$$\forall x, y \geq \quad \text{such that } |x-y| < \delta, \qquad \text{we have} \quad |\sqrt{x} - \sqrt{y}| < \varepsilon.$$

This proves that $x \mapsto \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

As for $f(x) = x^2$, it is not uniformly on $[0, \infty)$. For instance, consider $x_n = n$ and $y_n = n + \frac{1}{n}$. Then $|x_n - y_n| \to 0$, but

$$|f(x_n) - f(y_n)| = 2 + \frac{1}{n^2} \to 2 \neq 0.$$

Note: If you want to prove $\left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x-y|}$, you can assume that $x \geq y$, then

$$\left(\sqrt{x} - \sqrt{y}\right)^2 = x + y - 2\sqrt{xy} \leq x + y - 2\sqrt{y \cdot y} = x - y = |x - y|.$$

Which yields the required inequality after taking the square root of both sides. The case $y \geq x$ can be done similarly or by rearranging x and y in the inequality above.

Note: The square root function is not a Lipschitz function, but it is still uniformly continuous. This does not contradict Problem 4 since we never said that "all uniformly continuous functions are Lipschitz continuous", we only said that "all Lipschitz continuous functions are uniformly continuous".