Homework 4

Note: Always justify your answers.

Problem 1 (20pts)**.** Find the domain of convergence for the following power series:

(a) $\sum_{n=0}^{\infty} \frac{x^n}{n}$

(b) $\sum_{n=1}^{\infty} nx^n$

(c) $\sum_{n=0}^{\infty} n!x^n$

(d) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt[n]{n}}$

Problem 2 (20 pts)**.** Let $(a_n)_n$ and $(b_n)_n$ be two positive sequences such that

$$\lim_n \frac{a_n}{b_n} = \ell \in \mathbb{R}\backslash\{0\}.$$

Show that the series $\sum_n a_n$ converges if and only if the series $\sum_n b_n$ converges.

Problem 3 (20 pts)**.** Let $f : [0,1] \to [0,1]$ be a continuous function. Show that the equation $f(x) = x$ has at least one solution.

Problem 4 (20 pts)**.** Let $f : X \to Y$ be a function from a metric space (X, d_X) to a metric space (Y, d_Y) (meaning that d_X and d_Y are the metrics on X and Y, respectively). We say that f is *Lipschitz continuous* if there exists a constant $L > 0$ such that for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leq L d_X(x_1, x_2).$$

Show that if f is Lipschitz continuous, then it is continuous (you can also show that it is uniformly continuous if you want to).

Problem 5 (20 pts)**.** Is $f : [0,\infty) \to [0,\infty)$ defined by $f(x) = \sqrt{x}$ uniformly continuous? Justify your answer.

Same question for $f(x) = x^2$.