Homework 3

Note: Always justify your answers.

Problem 1 (15 points)**.** Let $(a_n)_n$ and $(b_n)_n$ two real sequences whose values are in $[0,1]$ and satisfy $a_n b_n \to 1$. Prove that $a_n \to 1$ and $b_n \to 1$.

Solution 1. We have $a_n b_n \leq a_n \leq 1$ since $b_n \in [0,1]$ (for all $n \in \mathbb{N}$). Therefore, by the squeeze theorem (proved in the recitation), we have $a_n \to 1$. The same argument leads to $b_n \to 1$.

Problem 2 (15 points)**.** We say that two sequences $(a_n)_n$ and $(b_n)_n$ are adjacent if

- One of them is increasing (e.g. a_n is increasing).
- The other one is decreasing (e.g. b_n is decreasing).
- The limit of the difference is zero (i.e. $a_n - b_n \to 0$).

For Example, the sequences $a_n = 1 - \frac{1}{n}$ and $b_n = 1 + \frac{1}{n}$ are adjacent.
Now, let $(a_n)_n$ and $(b_n)_n$ be two adjacent sequences. Prove that they converge to the same number.

Solution 2. First, we will show that b_1 is an upper bound of $\{a_n\}_n$. It would follow from that $(a_n)_n$ converges, and then $(b_n)_n$ converges.
Assume (for the sake of contradiction) that b_1 is not an upper bound of $\{a_n\}_n$, then there is $m \in \mathbb{N}$ such that $a_m > b_1$. Since $(a_n)_n$ is increasing, this would mean that $a_n > b_1$ for all $n > m$. This also means that $a_n > b_n$ for all $n > m$ because $(b_n)_n$ is decreasing.
So, for any $n > m$, we have $|a_n - b_n| = a_n - b_n \geq a_{m+1} - b_{m+1} > 0$. This contradicts $a_n - b_n \to 0$ (think of taking $\varepsilon = \frac{a_{m+1} - b_{m+1}}{2}$). Hence, b_1 is an upper bound of $\{a_n\}_n$. In fact, the same argument could be applied to any member of $(b_n)_n$, not just b_1.
Since $(a_n)_n$ is increasing and bounded above, it converges to some ℓ. The last condition implies that $b_n \to \ell$ too since

$$b_n = a_n - (a_n - b_n) \to \ell - 0 = \ell$$

Problem 3 (15 points)**.** Let $(a_n)_n$ be real sequence such that the subsequences $(a_{2n})_n$, $(a_{2n+1})_n$ and $(a_{3n})_n$ converge to ℓ_1, ℓ_2 and ℓ_3, respectively. Prove that $\ell_1 = \ell_2 = \ell_3$ and that $(a_n)_n$ converges.

Solution 3. Notice that $(a_{6n})_n$ a subsequence of $(a_{2n})_n$, which means that $a_{6n} \to \ell_1$. At the same time, it (referring to $(a_{6n})_n$) is a subsequence of $(a_{3n})_n$, which means that $a_{6n} \to \ell_3$. Since the limit is unique (Theorem 3.2.b), then $\ell_1 = \ell_3$.
To prove that $\ell_2 = \ell_3$, we consider the subsequence $(a_{6n+3})_n$ which is a subsequence of (a_{3n}) and a subsequence of $(a_{2n+1})_n$. Following the same idea as before, we get $\ell_2 = \ell_3$.

Now that we have shown that these three convergent subsequences have the same limit, let $\ell = \lim a_{2n} = \lim a_{2n+1} = \lim a_{3n}$. We will show that $a_n \to \ell$.
Let $\varepsilon > 0$, then there are $N_1, N_2 \in \mathbb{N}$ such that

$$\text{If } n \geq N_1, \text{ then } |a_{2n} - \ell| < \varepsilon, \text{ and if } n \geq N_2, \text{ then } |a_{2n+1} - \ell| < \varepsilon, \tag{1}$$

Now, let $N = \max(N_1, N_2)$. Then for any $n > 2N$, we have $|a_n - \ell| < \varepsilon$. This proves that $a_n \to \ell$.

Note: Notice that I used $2N$ instead of N. Since $|a_{N+1} - \ell| < \varepsilon$ may not be true.

Problem 4 (15 points)**.** Give an example of a real sequence $(a_n)_n$ such that $a_{n+1} - a_n \to 0$ and $(a_n)_n$ does not converge. (When you provide and example, justify why $a_{n+1} - a_n \to 0$ and why $(a_n)_n$ is not convergent).

Solution 4. Let $a_n = \sqrt{n}$, then

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n} + \sqrt{n+1}} \to 0. \tag{2}$$

But, $(a_n)_n$ does not converge since $a_n \to \infty$.

We can be more precise: To show that (2) holds, let $b_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$. Let $\varepsilon > 0$, then by the Archimedean property, there is $N \in \mathbb{N}$ such that $N > \varepsilon^{-2}$, which means that for all $n > N$, we have

$$0 \leq a_{n+1} - a_n = \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} \leq \varepsilon.$$

This shows that $a_{n+1} - a_n \to 0$.
To show that $a_n \to \infty$, we can the Archimedean property again: Let $M > 0$, then there is $N \in \mathbb{N}$ such that $N > M^2$. Now, for any $n \geq N$, we have

$$a_n \geq \sqrt{N} \geq M.$$

Problem 5 (15 points). Let $(a_n)_n$ be a real sequence, and let $(b_n)_n$ be the sequence

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n},$$

- Prove that if $a_n \to \ell \in \mathbb{R}$, then $b_n \to \ell$.
- Find an example where a_n diverges (does not converge) and b_n converges.

Solution 5.

- Assume $a_n \to \ell \in \mathbb{R}$. Let $\varepsilon > 0$, then there is $N_1 \in \mathbb{N}$ such that $|a_n - \ell| < \frac{\varepsilon}{2}$ for all $n > N_1$. Then, for any $n > N_1$, we have by the triangle inequality

$$\begin{aligned}
|b_n - \ell| &< \frac{|a_1 - \varepsilon| + |a_2 - \varepsilon| + \ldots |a_{N_1} - \varepsilon|}{n} + \frac{|a_{N_1+1} - \ell| + |a_{N_1+2} - \ell| + \ldots |a_n - \ell|}{n} \\
&\leq \frac{|a_1| + |a_2| + \cdots + |a_{N_1}|}{n} + \frac{N_1|\ell|}{n} + \frac{|a_{N_1+1} - \ell| + |a_{N_1+2} - \ell| + \ldots |a_n - \ell|}{n} \\
& \qquad \text{(Triangle inequality)} \\
&\leq \frac{|a_1| + |a_2| + \cdots + |a_{N_1}|}{n} + \frac{N_1|\ell|}{n} + \frac{(n - N_1)\varepsilon}{2n} \qquad (|a_n - \ell| < \varepsilon/2 \text{ for } n > N_1) \\
&\leq \frac{|a_1| + |a_2| + \cdots + |a_{N_1}|}{n} + \frac{\varepsilon}{2} + \frac{N_1(\varepsilon + 2|\ell|)}{2n}. \qquad \text{(Simplification)}
\end{aligned}$$

  Since $(a_n)_n$ converges, it is bounded, which means that there is $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Hence, we have

$$|b_n - \ell| \leq \frac{MN_1}{n} + \frac{N_1(\varepsilon + 2|\ell|)}{2n} + \frac{\varepsilon}{2}, \qquad \forall n \geq N_1.$$

  Now, let $N_2 \in \mathbb{N}$ be such that $N_2 > \frac{4MN_1}{\varepsilon}$ and let N_3 be such that $N_3 > \frac{2(\varepsilon+2|\ell|)N_1}{\varepsilon}$, and let $N = \max(N_1, N_2, N_3)$. Then, for any $n > N$, we have

$$|b_n - \ell| \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.$$

  This proves that $b_n \to \ell$.

- Consider $a_n = (-1)^n$, then $b_n \to 0$ since $\frac{-2}{n} \leq b_n \leq \frac{2}{n}$, but a_n does not converge.

Problem 6 (15 points). Let $(a_n)_n$ be a real sequence that satisfies the following condition

$$|a_{n+2} - a_{n+1}| < q|a_{n+1} - a_n|, \qquad \forall n \in \mathbb{N},$$

for some $q \in (0, 1)$. Show that $(a_n)_n$ converges.

Solution 6. Let $n \in \mathbb{N}$, then

$$|a_{n+1} - a_n| \leq q|a_n - a_{n-1}| \leq \cdots \leq q^{n-1}|a_2 - a_1|.$$

So, $|a_{n+1} - a_n| \leq Cq^n$ where $C = \frac{|a_2 - a_1|}{q}$. Since $q \in (0, 1)$, we know that $(a_n)_n$ is a Cauchy sequence (It follows from problem 8 in the recitation sheet). Hence, $(a_n)_n$ converges.

Note: To prove that $(a_n)_n$ is Cauchy, you can use the triangle inequality to show that

$$|a_{n+m} - a_n| \leq C\frac{q^n}{1 - q}.$$

Problem 7 (10 points)**.** Let $(a_n)_n$ and $(b_n)_n$ be two real sequences. Show that

$$\limsup_n \max(a_n, b_n) = \max\left(\limsup_n a_n, \limsup_n b_n\right)$$

Solution 7. To simplify the presentation, let

$$\ell_1 = \limsup_n a_n, \qquad \ell_2 = \limsup_n b_n, \qquad \ell_3 = \limsup_n \max(a_n, b_n),$$

Since $\max(a_n, b_n) \geq a_n$, we get $\ell_3 \geq \ell_1$. Similarly, we have $\ell_3 \geq \ell_1$. Thus, when we put those two inequalities together, we get

$$\ell_3 \geq \max(\ell_1, \ell_2).$$

Now, we need to prove the other inequality. To do that, we will use the "*epsilon of room*" trick. First, we assume, without loss of generality, that $\ell_1 \geq \ell_2$, and let $\varepsilon > 0$. Recall that

$$\ell_1 = \inf_{m \geq 0} \sup_{n \geq m} a_n.$$

Then, $\ell + \varepsilon$ is not a lower bound of $(\sup_{n \geq m} a_n)_m$, which means that there is $m_1 \in \mathbb{N}$ such that

$$a_n < \ell_1 + \varepsilon \qquad \forall n > m_1$$

The same holds for $(b_n)_n$ since $\ell_1 + \varepsilon > \ell_2$: There is $m_2 \in \mathbb{N}$ such that $b_n < \ell_1 + \varepsilon$ for all $n > m_2$. Now, let $m = \max(m_1, m_2)$, then for any $n > m$, we have

$$\max(a_n, b_n) \leq \ell_1 + \varepsilon$$

This implies that

$$\ell_3 \leq \max(\ell_1, \ell_2) + \varepsilon.$$

Since this is true for any ε, we get $\ell_3 \leq \max(\ell_1, \ell_2)$.

Note: There are other approaches to prove the inequality $\ell_3 \leq \max(\ell_1, \ell_2)$. For example, if you consider $c_n = \max(a_n, b_n)$, then there is a subsequence $(c_{n_k})_k$ that converges to ℓ_3 (we did not cover this in class, but it can be proved using the usual limit point ideas). Now, let us examine the terms of this subsequence: Let

$$A = \{k \in \mathbb{N} \mid a_{n_k} \geq b_{n_k}\}, \qquad B = \{k \in \mathbb{N} \mid a_{n_k} \leq b_{n_k}\}.$$

Then at least one of these sets is infinite. Say A is infinite, then there is a subsequence $\left(c_{n_{k_i}}\right)_i$ such that $c_{n_{k_i}} = a_{n_{k_i}}$, but this means that there is a subsequence of $(a_n)_n$ that converges to ℓ_3, which means that $\ell_3 \leq \ell_1$. Similarly, if B is infinite, we get $\ell_3 \leq \ell_2$. Either way, we have $\ell_3 \leq \max(\ell_1, \ell_2)$.