

# HOMEWORK 3

**Note:** Always justify your answers.

**Problem 1** (15 points). Let  $(a_n)_n$  and  $(b_n)_n$  two real sequences whose values are in  $[0, 1]$  and satisfy  $a_n b_n \rightarrow 1$ . Prove that  $a_n \rightarrow 1$  and  $b_n \rightarrow 1$ .

**Problem 2** (15 points). We say that two sequences  $(a_n)_n$  and  $(b_n)_n$  are adjacent if

- One of them is increasing (e.g.  $a_n$  is increasing).
- The other one is decreasing (e.g.  $b_n$  is decreasing).
- The limit of the difference is zero (i.e.  $a_n - b_n \rightarrow 0$ ).

For Example, the sequences  $a_n = 1 - \frac{1}{n}$  and  $b_n = 1 + \frac{1}{n}$  are adjacent.

Now, let  $(a_n)_n$  and  $(b_n)_n$  be two adjacent sequences. Prove that they converge to the same number.

**Problem 3** (15 points). Let  $(a_n)_n$  be real sequence such that the subsequences  $(a_{2n})_n$ ,  $(a_{2n+1})_n$  and  $(a_{3n})_n$  converge to  $\ell_1$ ,  $\ell_2$  and  $\ell_3$ , respectively. Prove that  $\ell_1 = \ell_2 = \ell_3$  and that  $(a_n)_n$  converges.

**Problem 4** (15 points). Give an example of a real sequence  $(a_n)_n$  such that  $a_{n+1} - a_n \rightarrow 0$  and  $(a_n)_n$  does not converge. (When you provide an example, justify why  $a_{n+1} - a_n \rightarrow 0$  and why  $(a_n)_n$  is not convergent).

**Problem 5** (15 points). Let  $(a_n)_n$  be a real sequence, and let  $(b_n)_n$  be the sequence

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n},$$

- Prove that if  $a_n \rightarrow \ell \in \mathbb{R}$ , then  $b_n \rightarrow \ell$ .
- Find an example where  $a_n$  diverges (does not converge) and  $b_n$  converges.

**Problem 6** (15 points). Let  $(a_n)_n$  be a real sequence that satisfies the following condition

$$|a_{n+2} - a_{n+1}| < q|a_{n+1} - a_n|, \quad \forall n \in \mathbb{N},$$

for some  $q \in (0, 1)$ . Show that  $(a_n)_n$  converges.

**Problem 7** (10 points). Let  $(a_n)_n$  and  $(b_n)_n$  be two real sequences. Show that

$$\limsup_n \max(a_n, b_n) = \max \left( \limsup_n a_n, \limsup_n b_n \right)$$