Homework 2

Note: Always justify your answers.

Problem 1 (10 points)**.** Show that $(0,1)$ is open and that $[0,1]$ is closed using the definition (both sets are in the metric space $\mathbb{R}$ with the Euclidean distance).

Note: The definitions are: E is open if every $x \in E$ is an interior point. E is closed if it contains all of its limit points.

Note: Do not use a theorem (like "Every open ball is open").

Solution 1.

- Let $x \in (0,1)$. Let $r = \frac{1}{2}\min(x, 1-x)$. Then, $B(x,r) \subset (0,1)$, which proves that $(0,1)$ is open.
- Let x be a limit point of $[0,1]$. If $x > 1$, then the ball $B(x, x-1)$ does not intersect $[0,1]$ (Contradiction). The same holds if $x < 0$. Therefore, if x is a limit point of $[0,1]$, then $x \in [0,1]$. This proves that $[0,1]$ contains all of its limit points.

Problem 2 (10 points)**.** Let $E = \{(x,y) \in \mathbb{R}^2 \mid y \neq x^2\}$. Show that E is open in $\mathbb{R}^2$ (with the Euclidean distance). Is E closed?

Solution 2. Notice that $E = E_1 \cup E_2$, where

$$E_1 = \{(x,y) \in \mathbb{R}^2 \mid y > x^2\}, \qquad E_2 = \{(x,y) \in \mathbb{R}^2 \mid y < x^2\}.$$

It is enough to show that both E_1 and E_2 are open since the union of open sets is open.

Let us start with E_1, consider a point $(a,b) \in E_1$ and let us attempt to find $\varepsilon > 0$ such that $B((a,b),\varepsilon) \subset E_1$. That is we want to show that $y > x^2$ whenever $(x-a)^2 + (y-b)^2 < \varepsilon^2$ (if this ε was chosen well).

Notice that

$$\begin{aligned} y - x^2 &= (y-b) + (a^2 - x^2) + (b - a^2) \\ &< (b - a^2) - |y-b| - |a-x||a+x| \end{aligned}$$

We have $|y-b| < \varepsilon$ and $|a-x| < \varepsilon$ because $(x-a)^2 + (y-a)^2 < \varepsilon$. Moreover, we have

$$|a+x| = |x - a + 2a| \leq |x-a| + 2|a| < \varepsilon + 2|a|,$$

so, we have

$$y - x^2 > (b - a^2) - \varepsilon - \varepsilon^2 - 2|a|\varepsilon$$

Assume that $\varepsilon < 1$, then $y - x^2 > (b-a^2) - 2\varepsilon(1+|a|)$. The right hand side would be positive if $\varepsilon < \frac{b-a^2}{2(1+|a|)}$.

In conclusion, if $(a,b) \in E_1$, then $B((a,b),\varepsilon) \subset E_1$, where

$$\varepsilon = \min\left(1, \frac{b-a^2}{4(1+|a|)}\right) > 0.$$

Proving that E_2 is open follows a similar idea with $\varepsilon < \min(1, \frac{a^2-b}{2(1+|a|)})$. Therefore, E is open.

E is not closed because $(0,0) \notin E$ is limit point. To see that, consider any ball $B((0,0),r)$ with $r > 0$, then it contains $(0, r/2) \in E$.

Problem 3 (10 points)**.** Is every point of every open set (with respect to the Euclidean distance) $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

Solution 3.

- Yes. Let $E \subseteq \mathbb{R}^2$ be open. Then, there exists $r > 0$ such that $B(x, r) \subset E$. Now, let $s > 0$, then the intersection $B(x, s) \cap B(x, r) = B(x, \min(s, r))$ is contained in E and contains infinitely many elements. Therefore, x is a limit point of E.
- No. A set with one element $\{x\}$ is closed, but it does not have limit point in $\mathbb{R}^2$.

Problem 4 (10 points)**.** Let X be a metric space with distance function d. Let $f : [0, \infty) \to [0, \infty)$. Let f be a function that satisfies the following conditions:

- $f(0) = 0$.
- f is strictly increasing, meaning that $\forall x, y \geq 0$, if $x < y$ then $f(x) < f(y)$.
- f is sub-additive, meaning that $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$.

Show that $f \circ d$ is a distance function on X (You need to show that it satisfies the three conditions in the definition of a metric space).

Note: This means that given one distance function on X, we can build many other distances.

Solution 4. Let $\tilde{d} = f \circ d$. Then

(0) d is non-negative because $\forall x, y \in X$, we have $d(x, y) \geq 0$, and then $\tilde{d}(x, y) = f(d(x, y)) \geq f(0) = 0$.

(1) Notice that the "strictly increasing" condition implies that if $f(x) = 0$ then $x = 0$. This means that if $\tilde{d}(x, y) = 0$, then $d(x, y) = 0$. Consequently, $x = y$.

(2) $\tilde{d}(y, x) = f(d(y, x)) = f(d(x, y)) = \tilde{d}(x, y)$ for any $x, y \in X$.

(3) Since f is increasing and d is a distance, we have

$$\tilde{d}(x, z) = f(d(x, z)) \leq f(d(x, y) + d(y, z))$$

Now, we use the sub-additivity of f to get

$$\tilde{d}(x, z) \leq f(d(x, y)) + f(d(y, z)) = \tilde{d}(x, y) + \tilde{d}(y, z)$$

Problem 5 (10pts)**.** Give an example of an open cover of the open interval $(0,1)$ with no finite subcover.

Solution 5. Consider $G = \{G_i\}_{i\in\mathbb{N}}$ defined as

$$G_i = \left(\frac{1}{i}, 1 - \frac{1}{i}\right)$$

Then,

- G is an open cover of $(0,1)$. Since given any $x \in (0,1)$, $x \in G_{i_0}$ where $i_0 > \max(\frac{1}{x}, \frac{1}{1-x})$. Note that a such i_0 exists by the Archimedean property.
- G has no finite subcover: Assume (for the sake of contradiction) that G admits a finite subcover, then $(0,1) \subset G_{i_1} \cup G_{i_2} \cup \cdots \cup G_{i_n}$ for some $n \in \mathbb{N}$ and $i_1, i_2, \ldots, i_n \in \mathbb{N}$. Let i_0 be the maximum of $i_1, i_2, \ldots, i_n$, then $(0,1) \subset G_{i_0}$.

  This means that $\frac{1}{2i_0} \in G_{i_0}$, and consequently $\frac{1}{2i_0} \geq \frac{1}{i_0}$ which leads to $1 \geq 2$, which is absurd. Hence, G has no finite subcover.

Problem 6 (20 points)**.** Let X be a metric space and $E \subset X$. Let E° be the set of all interior points of E, and recall that $\overline{E}$ is the closure of E (that is, the union of E and its limit points).

1. Show that E° is open.
2. Show that $(E^\circ)^c = \overline{E^c}$.
3. Do E and $\overline{E}$ have the same interior points?
4. Do E and E° have the same closure?

Solution 6.

1. Let $x \in E^\circ$, then there is $r > 0$ such that $B(x,r) \subset E$. Now, let $y \in B(x,r)$. Since $B(x,r)$ is open (this follows from the triangle inequality), there is $s > 0$ such that $B(y,s) \subset B(x,r) \subset E$. Then, $y \in E^\circ$. Since this holds for all $y \in B(x,r)$, we conclude that $B(x,r) \subset E^\circ$ and consequently E° is open.
2. First, we will show that $\overline{E^c} \subset (E^\circ)^c$: Let x be a point in the closure of E^c. Then, given any $r > 0$, there is $y \in B(x,r) \cap E^c$. Therefore, $x \notin E^\circ$, which means that $x \in (E^\circ)^c$.

   Next, we will show that $(E^\circ)^c \subset \overline{E^c}$: Let $x \in X$ not an interior point of E, then given any $r > 0$, there is $y \in B(x,r)$ such that $y \notin E$ (equivalently, $y \in E^c$). Hence, $x \in \overline{E^c}$.
3. No. Consider $E = (0,1) \cup (1,2)$ in $\mathbb{R}$ with the usual metric, then $E^\circ = E$ but $(\overline{E})^\circ = (0,2)$.
4. No, consider $E = \mathbb{Q}$ in $\mathbb{R}$ with the usual metric, then $E^\circ = \varnothing$ whereas the interior of $\overline{E}$ is $\mathbb{R}$.

Problem 7 (30 points)**.** Let $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be the following distance (called the discrete distance)

$$d(x,y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

1. Prove that d is a distance function (Show that it satisfies the three conditions in the definition).
2. (The following shows that the closure of the open ball is not necessarily the closed ball in an arbitrary metric space) Let $x \in \mathbb{R}$.
   (a) What is $B(x,1)$?
   (b) What is $\overline{B}(x,1)$?
   (c) What the closure of $B(x,1)$?
   (d) Do we have $\overline{B}(x,1) = \overline{B(x,1)}$?
3. Which subsets of $\mathbb{R}$ with the distance function d are open? Which ones are closed?
4. Which subsets of $\mathbb{R}$ with the distance function d are compact?

Solution 7.

1. d is non-negative and $d(x,y) = 0$ implies $x = y$ from the definition, the symmetry also follows from the definition. To prove the triangle inequality, consider $x, y, z \in \mathbb{R}$: If $x = z$, then $d(x,z) \leq d(x,y) + d(y,z)$. If $x \neq z$, then either $y \neq z$ or $y \neq x$, either way $d(x,z) \leq d(x,y) + d(y,z)$.
2.
   (a) $B(x,1)$ is the set of all points y such that $d(x,y) < 1$. Since this only holds for $y = x$, we have $B(x,1) = \{x\}$.
   (b) $\overline{B}(x,1) = \mathbb{R}$ since for any $y \in \mathbb{R}$, we have $d(x,y) \leq 1$.
   (c) Let y be a limit point of $B(x,1)$, then $B(x,1) \cap B(y,1)$ contains an element different from y, which is impossible since $B(y,1)$. Hence, $B(x,1)$ has no limit points and $\overline{B(x,1)} = \{x\}$.
   (d) Clearly not, from the previous two points. This is an example of a metric where the closure of the open ball is not the closed ball (common misconception).
3. Any subset of $\mathbb{R}$ is open with this metric since given any $E \subset \mathbb{R}$ and $x \in E$, the ball $B(x,1) \subset E$. Also, any set is closed: Given a set $E \subset \mathbb{R}$, we know that E^c is open. Hence, E is closed (Theorem 2.23).
4. In this metric, a set is compact iff it is finite:
   - If a set E is finite then it must be compact (this applies to any metric space): Let $G = \{G_i\}_{i \in I}$ be an open cover of $E = \{x_1, x_2, \ldots, x_n\}$, then for each $1 \leq k \leq n$, there is $i_k \in I$ such that $x_k \in G_{i_k}$ because G is a cover of E. This means that

$$E \subset G_{i_1} \cup G_{i_2} \cup \ldots G_{i_n}.$$

- If E was infinite, let $G_x = \{x\}$, then $G = \{G_x\}_{x \in E}$ is an open cover of E but G has no finite subcover.