Homework 2

Note: Always justify your answers.

Problem 1 (10 points)**.** Show that $(0,1)$ is open and that $[0,1]$ is closed using the definition (both sets are in the metric space $\mathbb{R}$ with the Euclidean distance).
Note: The definitions are: E is open if every $x \in E$ is an interior point. E is closed if it contains all of its limit points.
Note: Do not use a theorem (like "Every open ball is open").

Problem 2 (10 points)**.** Let $E = \{(x,y) \in \mathbb{R}^2 \mid y \neq x^2\}$. Show that E is open in $\mathbb{R}^2$ (with the Euclidean distance). Is E closed?

Problem 3 (10 points)**.** Is every point of every open set (with respect to the Euclidean distance) $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

Problem 4 (10 points)**.** Let X be a metric space with distance function d. Let $f : [0,\infty) \to [0,\infty)$. Let f be a function that satisfies the following conditions:

- $f(0) = 0$.
- f is strictly increasing, meaning that $\forall x, y \geq 0$, if $x < y$ then $f(x) < f(y)$.
- f is sub-additive, meaning that $f(x+y) \leq f(x) + f(y)$ for all $x, y \geq 0$.

Show that $f \circ d$ is a distance function on X (You need to show that it satisfies the three conditions in the definition of a metric space).
Note: This means that given one distance function on X, we can build many other distances.

Problem 5 (10pts)**.** Give an example of an open cover of the open interval $(0,1)$ with no finite subcover.

Problem 6 (20 points)**.** Let X be a metric space and $E \subset X$. Let E° be the set of all interior points of E, and recall that $\overline{E}$ is the closure of E (that is, the union of E and its limit points).

- Show that E° is open.
- Show that $(E^\circ)^c = \overline{E^c}$.
- Do E and $\overline{E}$ have the same interior points?
- Do E and E° have the same closure?

Problem 7 (30 points)**.** Let $d : \mathbb{R} \times \mathbb{R} \to [0,\infty)$ be the following distance (called the discrete distance)

$$d(x,y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

- Prove that d is a distance function (Show that it satisfies the three conditions in the definition).
- (The following shows that the closure of the open ball is not necessarily the closed ball in an arbitrary metric space) Let $x \in \mathbb{R}$.

  - What is $B(x,1)$?
  - What is $\overline{B}(x,1)$?
  - What the closure of $B(x,1)$?
  - Do we have $\overline{B}(x,1) = \overline{B(x,1)}$?

- Which subsets of $\mathbb{R}$ with the distance function d are open? Which ones are closed?
- Which subsets of $\mathbb{R}$ with the distance function d are compact?