Homework 1

Problem 1 (25 pts)**.** Let A be a nonempty subset of positive rational numbers such that $\inf(A) > 0$, and let

$$B = \left\{ \frac{1}{x} \mid x \in A \right\}.$$

Show that B is bounded above, and that $\sup(B) = \frac{1}{\inf(A)}$.

Solution 1. Let $r = \inf(A)$. We need to prove two statements:

(i) r^{-1} is an upper bound of B.

(ii) If there is an upper bound s of B, then $s \geq r^{-1}$.

To prove (i), recall that $r < a$ for any $a \in A$. Now, let $b \in B$, then $\exists a \in A$ such that $b = a^{-1}$. Since $r \leq a$, we get $r^{-1} \geq a^{-1} = b$ (see Prop 1.18.e). Therefore, r is an upper bound of B.

Now, assume that there is $s \in \mathbb{Q}$ such that $s \geq b$ for all $b \in B$. Then, by construction of B, we have $s \geq a^{-1}$ for any $a \in A$. But this means that s^{-1} is a lower bound of A since $s^{-1} \leq a$ for all $a \in A$. Since r is the greatest lower bound of A, we get $s^{-1} \leq r$. Equivalently, $s \geq r^{-1}$.

Important note: We have implicitly used the fact that $r > 0$ when we applied Prop. 1.18.e. If $\inf(A) = 0$, then $\sup(A) = \infty$, but we have to tweak our logic slightly: We only need to show that given any $n \in \mathbb{N}$, there is $b \in B$ such that $b \geq n$. One way to prove it is by contradiction.

Problem 2 (25 pts)**.** Let A and B be two non-empty subsets of $\mathbb{Q}$. Show that

$$\sup(A+B) = \sup A + \sup B, \tag{1}$$

$$\sup(A-B) = \sup A - \inf B. \tag{2}$$

Here, $A+B = \{x+y \mid x \in A, y \in B\}$ and $A-B = \{x-y \mid x \in A, y \in B\}$.

Note: Some of the quantities in (1) and (2) might be infinite, so these equalities only make sense in the extended number line (See section 1.23 in the book).

Hint: You may want to discuss the case where the supremums/infimums are finite, and the case where (at least) one of them is infinite separately. Also, think about using problem 15 from the recitation to your advantage.

Solution 2. We only need to prove (1), since (2) follows directly from Problem 15 in the recitation worksheet and (1).

To prove (1), first we consider the case where $\sup(A)$ and $\sup(B)$ are both finite. Consider $a \in A$ and $b \in B$. Then,

$$a + b \leq \sup(A) + \sup(B).$$

Therefore, $\sup(A) + \sup(B)$ is an upper bound of $A+B$. Now, let s be an upper bound of $A+B$. Then, $s \geq a+b$ for any $a \in A$ and $b \in B$. In particular, for any fixed $b \in B$, we have $s - b \geq a$ for any $a \in A$. This means that $s-b$ is an upper bound of A. Therefore, $s - b \geq \sup(A)$, or equivalently, $s - \sup(A) \geq b$. Since this is true for any $b \in B$, we get $s \geq \sup(A) + \sup(B)$. In conclusion, $\sup(A+B) = \sup(A) + \sup(B)$.

If $\sup(A) = \infty$, then that means for any $n \in \mathbb{N}$, there is $a \in A$ such that $a \geq n$. Now, fix $b \in B$ and let $n_0 \in \mathbb{N}$ be such that $n_0 \geq -b$ (Archimedean property). Now, given any $n \in \mathbb{N}$, there is $a \in A$ such that $a \geq n + n_0$. Therefore, $a + b \geq n + n_0 + b \geq n$. Since this is truth for any $n \in \mathbb{N}$, we get $\sup(A+B) = \infty$. A similar argument holds for the case where $\sup(B) = \infty$.

Problem 3 (25 pts)**.** Let A and B two non-empty sets, come up with formulas for $\sup(A \cup B)$ and $\sup(A \cap B)$ in terms of the supremums and minimums of A and B, and prove said formulas.

Hint: Consider a few special cases for A and B to see the *pattern*.

Solution 3. We can express $\sup(A \cup B)$ explicitly as $\max(\sup(A), \sup(B))$. To prove this, assume (without loss of generality) that $\sup(B) \geq \sup(A)$, let $r = \sup(B)$.

(i) Let $x \in A \cup B$. If $x \in A$, then $x \leq \sup(A) \leq r$. If $x \in B$, then $x \leq \sup(B) = r$. Either way, $x \leq r$. Therefore, r is an upper bound of $A \cup B$.

(ii) Let s be an upper bound of $A \cup B$, then s is an upper bound of B. Therefore, $s \leq r$ since r is the least upper bound of B. Hence, r is the least upper bound of $A \cup B$.

Now, when it comes to $\sup(A \cap B)$, you might think that it would be $\min(\sup(A), \sup(B))$, which is true in some cases (like when $A \subset B$). But this is not true always. In fact, $A \cap B$ may even be empty.

The best we can prove is that $\sup(A \cap B) \leq \min(\sup(A), \sup(B))$. This is true since both $\sup(A)$ and $\sup(B)$ are upper bounds of $A \cap B$:

$$\text{Given } x \in A \cap B,\ (x \in A \text{ and } x \in B)\,.\ \text{Therefore, }\ (x \leq \sup(A) \text{ and } x \leq \sup(B))\,.$$

In conclusion, if $\sup(A \cap B)$ exists, then it is less than or equal to $\min(\sup(A), \sup(B))$.

Problem 4 (25 pts)**.** Let A be given by

$$A = \left\{ r \in \mathbb{Q} \;\middle|\; r = \frac{m}{n} \text{ for some } m, n \in \mathbb{N} \text{ that satisfy } m < 2n \right\}$$

Find $\inf A$ and $\sup A$.

Solution 4. We have $\inf(A) = 0$ and $\sup(A) = 2$. Here, I will only prove the first one (the second follows a similar idea).

To prove that $0 = \inf(A)$, we need to show that 0 is a lower bound of A and that it is the greatest lower bound. First, we know that $0 \leq m/n$ for any integers m, n, then 0 is indeed a lower bound of A. To show that is the greatest lower bound. Assume that A admits another lower bound $\frac{p}{q} \in \mathbb{Q}$: If $\frac{p}{q} \leq 0$, we are done. If $p/q > 0$, then $\frac{1}{2q} \in A$ and $\frac{1}{2q} < \frac{p}{q}$. Therefore, $\frac{p}{q}$ is not a lower bound of A (Contradiction). In conclusion, 0 is the infimum of A.