Homework 1

Problem 1 (25 pts). Let A be a nonempty subset of positive rational numbers such that $\inf(A) > 0$, and let

$$B = \left\{ \frac{1}{x} \mid x \in A \right\}.$$

Show that B is bounded above, and that $\sup(B) = \frac{1}{\inf(A)}$.

Problem 2 (25 pts). Let A and B be two non-empty subsets of $\mathbb{Q}$. Show that

$$\sup(A+B) = \sup A + \sup B, \tag{1}$$

$$\sup(A-B) = \sup A - \inf B. \tag{2}$$

Here, $A + B = \{x + y \mid x \in A, y \in B\}$ and $A - B = \{x - y \mid x \in A, y \in B\}$.

Note: Some of the quantities in (1) and (2) might be infinite, so these equalities only make sense in the extended number line (See section 1.23 in the book).

Hint: You may want to discuss the case where the supremums/infimums are finite, and the case where (at least) one of them is infinite separately. Also, think about using problem 15 from the recitation to your advantage.

Problem 3 (25 pts). Let A and B two non-empty sets, come up with formulas for $\sup(A \cup B)$ and $\sup(A \cap B)$ in terms of the supremums and minimums of A and B, and prove said formulas.

Hint: Consider a few special cases for A and B to see the *pattern*.

Problem 4 (25 pts). Let A be given by

$$A = \left\{ r \in \mathbb{Q} \;\middle|\; r = \frac{m}{n} \text{ for some } m, n \in \mathbb{N} \text{ that satisfy } m < 2n \right\}$$

Find $\inf A$ and $\sup A$.