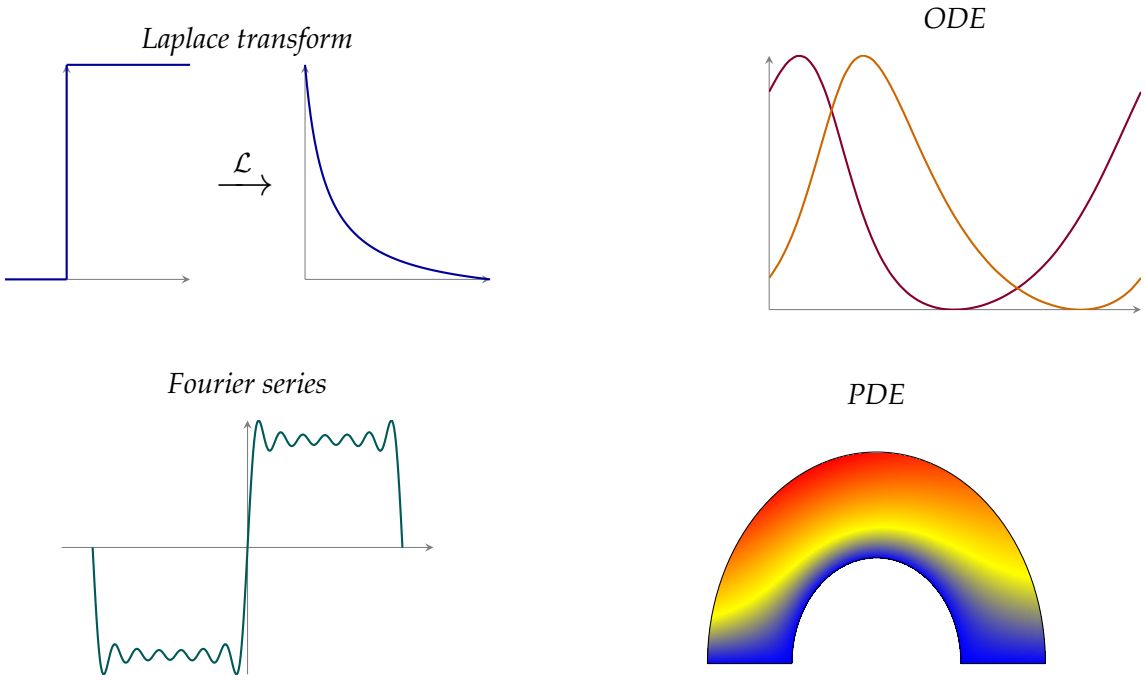
Problem book: Laplace transform, ODEs, Fourier series and PDEs

Haroun Meghaichi



This problem book is a collection of homework problems assigned in Math 4564 (Laplace transform, ODEs, Fourier series, and PDEs) at Virginia Tech between 2020 and 2022, together with worked solutions. It occasionally, though sparingly, references the course textbook:

D. G. Zill, *Differential Equations with Boundary-Value Problems*, 9th edition, Cengage.

Exercise 1. Without using the table, find the Laplace transform of $f(t) = t^2$ and $g(t) = t^3$. (Hint: use integration by parts).

Solution 1. We start with $\mathcal{L}\{f(t)\}$:

$$\begin{aligned} F(s) = \int_0^\infty \underbrace{t^2}_{u}\underbrace{e^{-st}}_{dv}\,dt &= \left[\frac{-t^2e^{-st}}{s}\right]_0^\infty + \frac{2}{s}\int_0^\infty te^{-st}\,dt \\ &= \frac{2}{s}\int_0^\infty \underbrace{t}_{u}\underbrace{e^{-st}}_{dv}\,dt \\ &= \frac{2}{s}\left(\left[\frac{-te^{-st}}{s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt\right) \\ &= \frac{2}{s^2}\int_0^\infty e^{-st}\,dt \\ &= \frac{2}{s^3}. \end{aligned}$$

Now, let us move to $\mathcal{L}\{g(t)\}$:

$$\begin{aligned} G(s) = \int_0^\infty \underbrace{t^3}_{u}\underbrace{e^{-st}}_{dv}\,dt &= \left[\frac{-t^3e^{-st}}{s}\right]_0^\infty + \frac{3}{s}\int_0^\infty t^2e^{-st}\,dt \\ &= \frac{3}{s}\int_0^\infty t^2e^{-st}\,dt \\ &= \frac{3}{s}\cdot\frac{2}{s^3} = \frac{6}{s^4}. \end{aligned}$$

Side note: There is a pattern here to be observed:

$$\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}$$

If you expand it further, you'd get

$$\mathcal{L}\{t^n\} = \frac{n}{s}\cdot\frac{n-1}{s}\mathcal{L}\left\{t^{n-2}\right\} = \frac{n}{s}\cdot\frac{n-1}{s}\cdots\frac{2}{s}\mathcal{L}\{t\} = \frac{n}{s}\cdot\frac{n-1}{s}\cdots\frac{2}{s}\cdot\frac{1}{s^2}$$

Which means that

$$\boxed{\mathcal{L}\{t^n\} = \frac{n(n-1)(n-2)\cdots 2}{s^{n+1}} = \frac{n!}{s^{n+1}}}$$

For any integer $n \geq 1$.

Exercise 2. Let

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t \le 2 \\ \sinh(t) & \text{if } t > 2 \end{cases}$$

Find $\mathcal{L}\{f(t)\}$.

Solution 2. We start by breaking down the LT integral into two integrals

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^2 e^{-st}\,dt + \int_2^\infty \sinh(t)e^{-st}\,dt$$

We compute the two integrals separately:

$$\int_0^2 e^{-st}\,dt = \frac{1-e^{-2s}}{s}.$$

To compute the second integral, we use the definition of $\sinh(t)$ which is $\frac{e^t-e^{-t}}{2}$:

$$\begin{aligned}
\int_2^\infty \sinh(t)e^{-st}\,dt &= \frac{1}{2}\int_2^\infty (e^t - e^{-t})e^{-st}\,dt \\
&= \frac{1}{2}\int_2^\infty e^t\cdot e^{-st}\,dt - \frac{1}{2}\int_2^\infty e^{-t}\cdot e^{-st}\,dt \\
&= \frac{1}{2}\int_2^\infty e^{-(s-1)t}\,dt - \frac{1}{2}\int_2^\infty e^{-(s+1)t}\,dt \\
&= \frac{1}{2}\left[\frac{-e^{-(s-1)t}}{s-1}\right]_2^\infty - \frac{1}{2}\left[\frac{-e^{-(s+1)t}}{s+1}\right]_2^\infty \\
&= \frac{1}{2}\left(\frac{e^{-2(s-1)}}{s-1} - \frac{e^{-2(s+1)}}{s+1}\right) \qquad \text{(Assuming } s>1\text{)}
\end{aligned}$$

Finally we have:

$$F(s) = \frac{1-e^{-2s}}{s} + \frac{1}{2}\left(\frac{e^{-2(s-1)}}{s-1} - \frac{e^{-2(s+1)}}{s+1}\right).$$

Alternative approach to the second integral: Let us assume that $s > 1$ and let us call the second integral I (see my comment at the end):

$$I = \int_2^\infty \sinh(t)e^{-st}\,dt$$

Apply IBP (integration by parts) to get

$$I = \left[\cosh(t)e^{-st}\right]_2^\infty + s\int_0^\infty \cosh(t)e^{-st}\,dt$$

Note that if $s > 1$, we have $\lim_{t\to\infty} \cosh(t)e^{-st} = 0$. Then,

$$I = -\cosh(2)e^{-2s} + s\int_2^\infty \cosh(t)e^{-st}\,dt$$

We apply IBP again, to get

$$I = -\cosh(2)e^{-2s} + s\Big[\sinh(t)e^{-st}\Big]_2^\infty + s^2\int_2^\infty \sinh(t)e^{-st}\,dt = -\cosh(2)e^{-2s} - s\sinh(2)e^{-2s} + s^2 I.$$

Now, we need to solve a simple linear equation to get I:

$$I = -\cosh(2)e^{-2s} - s\sinh(2)e^{-2s} + s^2 I$$

we have

$$\boxed{I = \frac{\cosh(2) + s\sinh(2)}{s^2 - 1}e^{-2s}.}$$

This might look a bit different that what we got before but they are actually the same. To see that, you can use PFD and write $\cosh(2)$ and $\sinh(2)$ in terms of exponentials.If you want to use Mathematica to verify that the two are the same, the best way to do it is to simplify the difference between the two expressions:

```
In[1]:= A=(Cosh[2]+s Sinh[2])/(s^2-1)Exp[-2s];
        B=1/2(Exp[-2(s-1)]/(s-1)-Exp[-2(s+1)]/(s+1));

Out[1]= 0
```

Exercise 3. Is it possible to find two continuous functions of exponential order f and g such that $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\} + 1$? Justify your answer.

Hint: Does this violate the theorem 7.1.3?

Solution 3. The answer is **NO**, we know that if f and g are continuous and of exponential order, then

$$\lim_{s\to\infty} F(s) = \lim_{s\to\infty} G(s) = 0$$

However, if $F(s) = G(s) + 1$, by taking the limit, we'd get $0 = 1$ which is absurd. Therefore, the statement $\mathcal{L}\{f(t)\} = \mathcal{L}\{f(t)\} + 1$ is wrong for any two continuous function f, g of exponential order.

Alternative but equivalent approach: We can rewrite the given equality as

$$\mathcal{L}\{f(t) - g(t)\} = 1$$

Notice that $f(t) - g(t)$ is continuous and of exponential order (*if you take the difference/sum of two functions of exponential order, the result is of exponential order as well*). Therefore, if I define, $h(t) = f(t) - g(t)$, I should get

$$\lim_{s\to\infty} H(s) = 0,$$

but $H(s) = 1$, Contradiction!.

Exercise 4. Find the inverse Laplace transform of the following functions

1. $\dfrac{s+2}{s^2}$.

2. $\dfrac{2s+1}{s^2+s}$.

Solution 4. 1.

$$\mathcal{L}^{-1}\left\{\frac{s+2}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = 1 + 2t.$$

2. First, we need to apply a partial fraction decomposition. There are (at least) two ways to do it:

   (a) We try to find a, b such that

   $$\frac{2s+1}{s^2+s} = \frac{2s+1}{s(s+1)} = \frac{a}{s} + \frac{b}{s+1} = \frac{(a+b)s+a}{s(s+1)}$$

   This leads to a simple system $\begin{cases} a+b=2 \\ a=1 \end{cases}$, the solution is $a = b = 1$.

   (b) With a simple rearrangement, we have

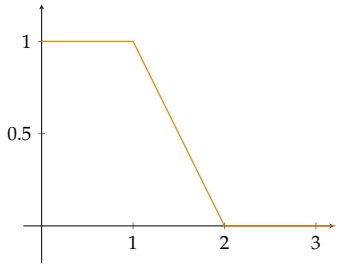
   $$\frac{2s+1}{s^2+s} = \frac{2s+1}{s(s+1)} = \frac{s+1}{s(s+1)} + \frac{s}{s(s+1)} = \frac{1}{s} + \frac{1}{s+1}$$

   Now, we go back to the ILT:

   $$\mathcal{L}^{-1}\left\{\frac{2s+1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = 1 + e^{-t}$$

Exercise 5. Consider the following function f:

$$f(t) = \begin{cases} 1, & 0 \le t \le 1, \\ (2-t), & 1 < t \le 2, \\ 0, & t > 2 \end{cases}$$



The graph of f.

Find $\mathcal{L}\{f(t)\}$. *Hint:* Break the integral into two.

Solution 5. Let us start with the integral:

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Since f is defined piecewise, we split the integral:

$$F(s) = \int_0^1 1e^{-st}dt + \int_1^2 (2-t)e^{-st}\,dt + \int_2^\infty 0e^{-st}\,dt$$

The last integral is zero. Now, you have two integrals to handle, I prefer computing them separately, start with the first one

$$\int_0^1 e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^1 = \frac{1-e^{-s}}{s}. \tag{1}$$

For the second one, we use integration by parts

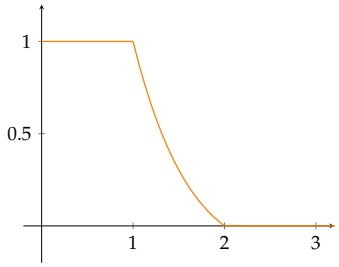
$$\begin{aligned} \int_1^2 (2-t)e^{-st}\,dt &= \left[(2-t)\frac{e^{-st}}{-s}\right]_1^2 - \int_1^2 (-1)\frac{e^{-st}}{-s}\,dt, \\ &= \frac{e^{-s}}{s} - \frac{1}{s}\int_1^2 e^{-st}\,dt \\ &= \frac{e^{-s}}{s} - \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_1^2 \\ &= \frac{e^{-s}}{s} + \frac{e^{-2s}-e^{-s}}{s^2}. \end{aligned} \tag{2}$$

Now, we add (1) and (2) together to get

$$\boxed{F(s) = \frac{1}{s} + \frac{e^{-2s}-e^{-s}}{s^2}}$$

Exercise 6. Consider the following function f:

$$f(x) = \begin{cases} 1, & 0 \le t \le 1, \\ (2-t)e^{1-t}, & 1 < t \le 2, \\ 0, & t > 2 \end{cases}$$



The graph of f.

Find $F(s)$. *Hint:* Break the integral into two.

Solution 6. Break the integral into

$$\int_0^\infty f(t)e^{-st}\,dt = \int_0^1 e^{-st}\,dt + \int_1^2 (2-t)e^{1-t}e^{-st}\,dt.$$

The first integral is easy

$$\int_0^1 e^{-st}\,dt = \frac{1-e^{-s}}{s}.$$

To evaluate the second integral, we use integration by part

$$\begin{aligned} \int_1^2 (2-t)e^{1-t}e^{-st}\,dt &= \int_1^2 (2-t)e^{1-(s+1)t}\,dt \\ &= \left[\frac{(2-t)e^{1-(s+1)t}}{-(s+1)}\right]_1^2 - \frac{1}{(s+1)^2}\int_1^2 e^{1-(s+1)t}\,dt \\ &= \frac{e^{-s}}{s+1} - \frac{e^{-s}-e^{-2s-1}}{(s+1)^2}. \end{aligned}$$

The final answer is

$$\frac{1-e^{-s}}{s} + \frac{e^{-s}}{s+1} - \frac{e^{-s}-e^{-2s-1}}{(s+1)^2}$$

Exercise 7. Let

$$f(t) = \begin{cases} 0 & \text{if } 0 \le t \le 3 \\ \cosh(t) & \text{if } t > 3 \end{cases}$$

Find $\mathcal{L}\{f(t)\}$.

Note 1: cosh is called the hyperbolic cosine function and defined as $\cosh(t) = \frac{1}{2}\left(e^t + e^{-t}\right)$.
Note 2: Since f is not exactly cosh, you cannot use the table directly so you'll have to start from the integral.

Solution 7. Since f is zero on $[0,3]$, we only need to integrate it from 3 to ∞:

$$F(s) = \int_3^\infty \cosh(t)e^{-st}\,dt$$

Now, we use the first note:

$$F(s) = \frac{1}{2}\int_3^\infty e^t e^{-st} + e^{-t}e^{-st}\,dt = \frac{1}{2}\int_3^\infty e^{(1-s)t} + e^{(-1-s)t}\,dt$$

We can integrate the two terms separately or together:

$$F(s) = \frac{1}{2}\left[\frac{e^{(1-s)t}}{1-s} + \frac{e^{(-1-s)t}}{-1-s}\right]_3^\infty = \frac{1}{2}\left(\frac{-e^{3(1-s)}}{1-s} - \frac{e^{3(-1-s)}}{-1-s}\right).$$

We can clean it up a bit:

$$F(s) = \frac{1}{2}\left(\frac{e^{3-3s}}{s-1} + \frac{e^{-3-3s}}{s+1}\right).$$

A different method: You can call the integral I, integrate by parts twice and solve for I:

$$\begin{aligned} I &= \int_3^\infty \cosh(t)e^{-st}\,dt \\ &= -\sinh(3)e^{-3s} + s\int_3^\infty \sinh(t)e^{-st}\,dt \\ &= -\sinh(3)e^{-3s} - s\cosh(3)e^{-3s} + s\int_3^\infty \cosh(t)e^{-st}\,dt \\ I &= -\sinh(3)e^{-3s} - s\cosh(3)e^{-3s} + s^2 I. \end{aligned}$$

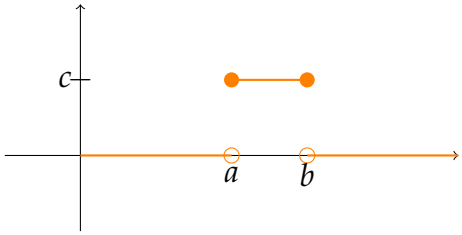
After solving for I, we find

$$I = \frac{s\cosh(3) + \sinh(3)}{s^2 - 1}.$$

Although that I may not look like the $F(s)$ that we have found, we can actually show that $I = F(s)$ (think about doing a PFD on I or you can write $F(s)$ as one fraction).

Exercise 8. Let a, b, c be positive real numbers such that $a < b$, find the Laplace transform of the following function

$$f(x) = \begin{cases} c, & \text{if } a \leq x \leq b, \\ 0, & \text{otherwise.} \end{cases}$$



The graph of f.

Food for thought (will not be graded): Would your answer change if $c < 0$? Would it change if a and/or b are negative?

Solution 8. We have

$$\begin{aligned} \int_0^\infty f(t)e^{-st}\,dt &= \int_0^a 0e^{-st}\,dt + \int_a^b ce^{-st}\,dt + \int_b^\infty 0e^{-st}\,dt \\ &= c\int_a^b e^{-st}\,dt \\ &= c\left(\frac{e^{-as} - e^{-bs}}{s}\right). \end{aligned}$$

Notes: The answer will not change if $c < 0$. However, if $a < 0 < b$, then

$$F(s) = \int_0^b ce^{-st}\,dt = c\left(\frac{1 - e^{-st}}{s}\right).$$

Also, if $b < 0$, then $F(s) = 0$ for all s.

Exercise 9. Let a, b be two real numbers such that $b > a > 0$, and let

$$f(t) = \begin{cases} 1, & a < t < b, \\ 0, & \text{otherwise} \end{cases}$$

Here's the graph of f (It might help you)

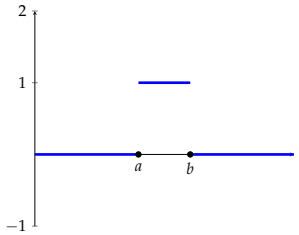


Figure 1: The graph of f

1. Write $f(t)$ as the combination of two Heaviside functions by filling the boxes below:

$$f(t) = \square\,\mathscr{U}\left(t - \square\right) + \square\,\mathscr{U}\left(t - \square\right)$$

2. Find the Laplace transform of f.

3. Use the first question (on this problem) and the 2nd translation theorem to compute the LT of

$$g(t) = \begin{cases} \sin(t), & \pi < t < 3\pi, \\ 0, & \text{otherwise} \end{cases}$$

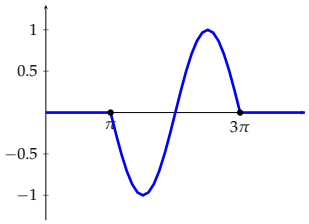


Figure 2: The graph of g

Note 1: You may need to use the following trig identity: $\sin(t - \pi) = -\sin(t)$.
Note 2: As you see here, a good manipulation of the Heaviside function can help you avoid 'nasty' integrals.

Solution 9. 1. $f(t) = \mathscr{U}(t - a) - \mathscr{U}(t - b)$. See the last example of 7.3.2.

2. $F(s) = \dfrac{e^{-as} - e^{-bs}}{s}$.

3. We have

$$g(t) = \sin(t)\mathscr{U}(t-\pi) - \sin(t)\mathscr{U}(t-3\pi) = -\sin(t-\pi)\mathscr{U}(t-\pi) + \sin(t-3\pi)\mathscr{U}(t-3\pi).$$

Apply the second translation theorem to get

$$G(s) = -\frac{1}{s^2+1}e^{-\pi s} + \frac{1}{s^2+1}e^{-3\pi s}.$$

Exercise 10. Let $n \in \mathbb{N}$ and

$$f_n(t) = \begin{cases} n & \text{if } t \in \left[0, \frac{1}{n}\right], \\ 0 & \text{if } t > \frac{1}{n}. \end{cases}$$

1. Find $F_n(s)$, the Laplace transform of $f_n(t)$. *Hint:* Break the LT integral into two integrals, you can use Mathematica at the end to verify your answer.
2. What is
$$\lim_{n\to\infty} F_n(s)?$$
(You can use the Limit command, De L'Hopital's rule, Taylor series or any other method).
3. Show that the previous limit is not the Laplace transform of a piecewise continuous function of exponential order.

Note: This shows that the limit of a sequence of Laplace transforms of "nice functions" is not necessarily a Laplace transform of a "nice function".

Solution 10. 1.

$$\begin{aligned} F_n(s) &= n\int_0^{\frac{1}{n}} e^{-st}\,dt + n\int_{\frac{1}{n}}^{\infty} 0\cdot e^{-st}\,dt \\ &= n\int_0^{\frac{1}{n}} e^{-st}\,dt \\ &= n\left[\frac{-e^{-st}}{s}\right]_0^{\frac{1}{n}} \\ &= \frac{n(1-e^{-\frac{s}{n}})}{s}. \end{aligned}$$

2. To compute the limit, we use the `Limit` command:

```
In[2]:= Limit[(n - E^(-(s/n)) n)/s, n->Infinity]

Out[2]= 1
```

If you want to compute it by hand, you can use the well known limit $\lim_{x\to 0}\frac{e^x-1}{x} = 1$:

$$\lim_{n\to\infty}\frac{n(1-e^{-\frac{s}{n}})}{s} = \lim_{n\to\infty}\frac{e^{-\frac{s}{n}}-1}{\frac{-s}{n}} = 1$$

(Note that $\frac{-s}{n} \to 0$ as $n \to \infty$). You can use L'Hopital's rule as well (details omitted).

3. Since

$$\lim_{s\to\infty} 1 = 1 \neq 0$$

Then $F(s) = 1$ is not the Laplace transform of piecewise continuous function of exponential order (Theorem 7.1.3).

Exercise 11. Find the Laplace transform of $f(t) = \cos\left(t - \frac{\pi}{3}\right)$.

Solution 11. Use the trig identity:

$$\boxed{\cos(a-b) = \sin(a)\sin(b) + \cos(a)\cos(b)}$$

to get

$$\begin{aligned}
F(s) &= \mathcal{L}\left\{\cos\left(t - \frac{\pi}{6}\right)\right\} \\
&= \mathcal{L}\left\{\sin(t)\sin\left(\frac{\pi}{3}\right) + \cos(t)\cos\left(\frac{\pi}{3}\right)\right\} \\
&= \mathcal{L}\left\{\frac{\sqrt{3}}{2}\sin(t) + \frac{1}{2}\cos(t)\right\} \\
&= \frac{\sqrt{3}}{2}\mathcal{L}\left\{\sin(t)\right\} + \frac{1}{2}\mathcal{L}\left\{\cos(t)\right\} \\
&= \frac{\sqrt{3}}{2}\cdot\frac{1}{s^2+1} + \frac{1}{2}\cdot\frac{s}{s^2+1} \\
&= \frac{\sqrt{3}+s}{2(s^2+1)}.
\end{aligned}$$

Exercise 12. Find the Laplace transform of $f(t) = \sin\left(t - \frac{\pi}{6}\right)$.

Solution 12. Use the trig identity:

$$\boxed{\sin(a-b) = \sin(a)\cos(b) - \sin(b)\cos(a)}$$

to get

$$\begin{aligned}
F(s) &= \mathcal{L}\left\{\sin\left(t - \frac{\pi}{6}\right)\right\} \\
&= \mathcal{L}\left\{\sin(t)\cos\left(\frac{\pi}{6}\right) - \cos(t)\sin\left(\frac{\pi}{6}\right)\right\} \\
&= \mathcal{L}\left\{\frac{\sqrt{3}}{2}\sin(t) - \frac{1}{2}\cos(t)\right\} \\
&= \frac{\sqrt{3}}{2}\mathcal{L}\left\{\sin(t)\right\} - \frac{1}{2}\cos(t) \\
&= \frac{\sqrt{3}}{2}\cdot\frac{1}{s^2+1} - \frac{1}{2}\cdot\frac{s}{s^2+1} \\
&= \frac{\sqrt{3}-s}{2(s^2+1)}.
\end{aligned}$$

Note: There is at least three other ways to do this:

1. You can integrate by parts twice, you just need to be careful about evaluating at zero correctly.
2. You can use complex numbers: evaluate $\int_0^\infty e^{i\left(t-\frac{\pi}{6}\right)}e^{-st}\,dt$, then take the imaginary part of the result (think about Euler identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$).
3. Make a u-sub: $u = t - \frac{\pi}{6}$ and use the antiderivative of $\sin(u)e^{-su}$ that we "kind of" derived in the notes.

Exercise 13. Find the inverse Laplace transform of the following functions

1. $\dfrac{1}{3s^2-18}$
2. $\dfrac{1}{s^2+s-20}$.
3. $\dfrac{1}{s^2+2s+1}$.
4. $\dfrac{1}{s^2+2s+2}$.
5. $\dfrac{1}{s^2+2s+5}$.
6. $\dfrac{e^{-3s}}{s^2+2s+5}$.
7. $\dfrac{1-3s^2-5s}{s(s+1)^2+s}$
8. $\dfrac{1-e^{-2s}}{s(s^2+1)}$
9. $\dfrac{1}{s^2+2s-15}$.
10. $\dfrac{1}{s^2+2s+10}$.
11. $\dfrac{1-3s^2-4s}{s(s+1)^2+s}$

Solution 13. 1. We have

$$\frac{1}{3s^2-18}=\frac{1}{3}\left(\frac{1}{s^2-6}\right)=\frac{1}{3\sqrt{6}}\left(\frac{\sqrt{6}}{s^2-(\sqrt{6})^2}\right)$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{1}{3s^2-18}\right\}=\frac{1}{3\sqrt{6}}\mathcal{L}^{-1}\left\{\frac{\sqrt{6}}{s^2-(\sqrt{6})^2}\right\}=\frac{1}{3\sqrt{6}}\sinh\left(t\sqrt{6}\right)$$

2. We apply PFD to the given fraction

$$\frac{1}{s^2+s-20}=\frac{1}{(s+5)(s-4)}=\frac{a}{s+5}+\frac{b}{s-4}=\frac{(a+b)s-4a+5b}{(s+5)(s-4)}.$$

Which leads to the system

$$\begin{cases} a+b=0, \\ -4a+5b=1, \end{cases} \overset{4(1)+(2)}{\Longrightarrow} 9b=1\Rightarrow b=\frac{1}{9}\Rightarrow a=\frac{-1}{9}.$$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+s-20}\right\}=-\frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}+\frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}=\frac{-e^{-5t}+e^{4t}}{9}.$$

3. We have

$$\frac{1}{s^2+2s+1} = \frac{1}{(s+1)^2}$$

Using the 1st translation theorem, we get

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} e^{-t} = te^{-t}$$

4. Complete the square

$$\frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1}$$

Apply the second translation theorem

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} e^{-t} = \sin(t)e^{-t}$$

5. Check example 3 for more details. We have

$$\frac{1}{s^2+2s+5} = \frac{1}{(s+1)^2+2^2}$$

Apply the second translation theorem

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\} e^{-t} = \frac{1}{2}\sin(2t)e^{-t}.$$

6. Use the previous ILT and the second translation theorem to get

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+1)^2+2^2}\right\} = \frac{1}{2}\sin(2(t-3))e^{-(t-3)}\mathscr{U}(t-3).$$

7. We need to factor the denominator first and then apply PFD

$$\begin{aligned} F(s) = \frac{1-3s^2-5s}{s(s+1)^2+s} = \frac{1-3s^2-5s}{s((s+1)^2+1)} &= \frac{a}{s} + \frac{bs+c}{(s+1)^2+1} \\ &= \frac{(a+b)s^2+(2a+c)s+2a}{s((s+1)^2+1)} \end{aligned}$$

By comparing the coefficients, we get

$$\begin{cases} 2a = 1 \\ 2a+c = -5 \\ a+b = -3 \end{cases} \implies \begin{cases} a = \frac{1}{2} \\ b = \frac{-7}{2} \\ c = -6. \end{cases}$$

Thus,

$$F(s) = \frac{1}{2s} + \frac{\frac{-7}{2}s - 6}{(s+1)^2+1} = \frac{1}{2s} + \frac{\frac{-7}{2}s - \frac{7}{2}}{(s+1)^2+1} + \frac{-\frac{5}{2}}{(s+1)^2+1}$$

So,

$$F(s) = \frac{1}{2s} - \frac{7}{2}\cdot\frac{s+1}{(s+1)^2+1} - \frac{5}{2}\cdot\frac{1}{(s+1)^2+1}$$

The inverse LT of the first term is easy, to the find the other two inverse LTs we need to use the first translation theorem.

$$f(t) = \frac{1}{2} - \frac{7}{2}\cos(t)e^{-t} - \frac{5}{2}\sin(t)e^{-t}.$$

8. You can always perform a PFD like we did in the previous example but there is a shortcut here

$$\frac{1}{s(s^2+1)} = s\frac{s^2}{s^2(s^2+1)} = s\frac{(s^2+1)-s^2}{s^2(s^2+1)} = s\frac{1}{s^2} - s\frac{1}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1}.$$

Now, we have

$$\frac{1-e^{-2s}}{s(s^2+1)} = \left(\frac{1}{s} - \frac{s}{s^2+1}\right) - \left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-2s}$$

The first part is easy, you need to apply the 2nd translation theorem to the second one to get

$$1 - \cos(t) - (1 - \cos(t-2))\,\mathscr{U}(t-2)$$

9. We have

$$\frac{1}{s^2+2s-15} = \frac{1}{(s+5)(s-3)} = \frac{a}{s+5} + \frac{b}{s-3} = \frac{(a+b)s + (5b-3a)}{(s+5)(s-3)}$$

We have $a+b=0$ and $5b-3a=1$ which leads to $a = \frac{-1}{8}$ and $b = \frac{1}{8}$. Therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s-15}\right\} = \frac{e^{3t} - e^{-5t}}{8}.$$

Note: There is another way to do this using the translation theorem:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s-15}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2-4^2}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{4}{(s+1)^2-4^2}\right\} = \frac{1}{4}\sinh(4t)e^{-t}.$$

10. We have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+3^2}\right\} = \frac{1}{3}\sin(3t)e^{-t}.$$

11. We need to factor the denominator first and then apply PFD

$$F(s) = \frac{1-3s^2-4s}{s(s+1)^2+s} = \frac{1-3s^2-4s}{s((s+1)^2+1)} = \frac{a}{s} + \frac{bs+c}{(s+1)^2+1}$$
$$= \frac{(a+b)s^2+(2a+c)s+2a}{s((s+1)^2+1)}$$

By comparing the coefficients, we get

$$\begin{cases} 2a = 1 \\ 2a + c = -4 \\ a + b = -3 \end{cases} \implies \begin{cases} a = \frac{1}{2} \\ b = \frac{-7}{2} \\ c = -5. \end{cases}$$

Thus,

$$F(s) = \frac{1}{2s} + \frac{\frac{-7}{2}s - 6}{(s+1)^2+1} = \frac{1}{2s} + \frac{\frac{-7}{2}s - \frac{7}{2}}{(s+1)^2+1} + \frac{-\frac{3}{2}}{(s+1)^2+1}$$

So,

$$F(s) = \frac{1}{2s} - \frac{7}{2}\cdot\frac{s+1}{(s+1)^2+1} - \frac{3}{2}\cdot\frac{1}{(s+1)^2+1}$$

The inverse LT of the first term is easy, to the find the other two inverse LTs we need to use the first translation theorem.

$$f(t) = \frac{1}{2} - \frac{7}{2}\cos(t)e^{-t} - \frac{3}{2}\sin(t)e^{-t}.$$

Exercise 14. Solve the following differential equations using Laplace transform:

1. $y'(t) + y(t) = \cos(3t), \qquad y(0) = 0.$
2. $y''(t) + y(t) = \cos(3t), \qquad y(0) = 1, y'(0) = 2.$
3. $y''(t) - 4y'(t) + 4y(t) = \cos(2t), \qquad y(0) = 0,\ y'(0) = 1.$
4. $y'(t) - y(t) = \cosh(3t), \qquad y(0) = 0.$
5. $y''(t) - 2y'(t) + 5y(t) = \cos(2t), \qquad y(0) = 0,\ y'(0) = 1.$

Solution 14. 1. There are three steps:

(a) Apply LT to both sides

$$sY(s) - 0 + Y(s) = \frac{s}{s^2+9}$$

(b) Solve for $Y(s)$:

$$Y(s) = \frac{s}{(s+1)(s^2+9)}$$

(c) Apply the ILT: First, start with the PFD

$$\frac{s}{(s+1)(s^2+9)} = \frac{a}{s+1} + \frac{bs+c}{s^2+9} = \frac{(a+b)s^2 + (b+c)s + 9a + c}{(s+1)(s^2+9)} \longrightarrow \begin{cases} a+b=0, \\ b+c=1, \\ 9a+c=0. \end{cases}$$

The solution to the above problem is $(\frac{-1}{10}, \frac{1}{10}, \frac{9}{10})$. Therefore,

$$Y(s) = \frac{1}{10}\left(\frac{-1}{s+1} + \frac{s}{s^2+9} + \frac{9}{s^2+9}\right).$$

Consequently,

$$y(t) = \frac{1}{10}\left(-e^{-t} + \cos(3t) + 3\sin(3t)\right).$$

2. We follow the same steps

(a) Apply LT to both sides

$$s^2Y(s) - s - 2 + Y(s) = \frac{s}{s^2+9}$$

(b) Solve for $Y(s)$

$$Y(s) = \frac{s}{(s^2+9)(s^2+1)} + \frac{s+2}{s^2+1}$$

(c) Apply the ILT: First, you need to apply the PFD to the first fraction

$$\frac{s}{(s^2+9)(s^2+1)} = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+9} = \frac{(a+c)s^3+(b+d)s^2+(9a+c)s+9b+d}{(s^2+9)(s^2+1)}$$

It is easy to see that $b = d = 0$. After that, you have to solve a 2×2 system to get $a = \frac{1}{8}$ and $b = \frac{-1}{8}$. Now, we have:

$$Y(s) = \frac{1}{8}\left(\frac{s}{s^2+1} - \frac{s}{s^2+9}\right) + \frac{s}{s^2+1} + \frac{2}{s^2+1}$$

Which gives us

$$y(t) = \frac{1}{8}\left(\cos(t) - \cos(3t)\right) + \cos(t) + 2\sin(t)$$

$$\boxed{y(t) = \frac{9}{8}\cos(t) - \frac{1}{8}\cos(3t) + 2\sin(t)}$$

3. I will skip the first few steps and jump right into

$$Y(s) = \frac{1}{(s-2)^2} + \frac{s}{(s-2)^2(s^2+4)}.$$

Therefore,

$$y(t) = te^{2t} + \mathcal{L}^{-1}\left\{\frac{s}{(s-2)^2(s^2+4)}\right\}$$

Apply PFD

$$\frac{s}{(s-2)^2(s^2+4)} = \frac{a}{s-2} + \frac{b}{(s-2)^2} + \frac{cs+d}{s^2+4}$$

You will get the following system

$$\begin{cases} a+c=0 \\ -2a+b-4c+d=0 \\ 4a+4c-4d=1 \\ -8a+4b+4d=0 \end{cases} \Longrightarrow \begin{cases} a=0 \\ b=\frac{1}{4} \\ c=0 \\ d=-\frac{1}{4}. \end{cases}$$

(A good first step would be using (1) and (3) to find d). Now,

$$\frac{s}{(s-2)^2(s^2+4)} = \frac{1}{4}\left(\frac{1}{(s-2)^2} - \frac{1}{s^2+4}\right)$$

Finally, we get

$$y(t) = te^{2t} + \frac{1}{4}te^{2t} - \frac{1}{8}\sin(2t) = \frac{5}{4}te^{2t} - \frac{1}{8}\sin(2t)$$

Note 1: You can combine the two fraction in the first line and work from there.

Note 2: You can avoid the system by noticing that

$$\frac{s}{(s-2)^2(s^2+4)} = \frac{1}{4}\frac{4s}{(s^2-4s+4)(s^2+4)} = \frac{1}{4}\frac{(s^2+4)-(s^2-4s+4)}{(s^2-4s+4)(s^2+4)}.$$

4. There are three steps:

   (a) Apply LT to both sides
   $$sY(s) - 0 - Y(s) = \frac{s}{s^2-9}$$

   (b) Solve for $Y(s)$:
   $$Y(s) = \frac{s}{(s-1)(s^2-9)}$$

   (c) Apply the ILT: First, start with the PFD

$$\frac{s}{(s-1)(s^2-9)} = \frac{a}{s-1} + \frac{bs+c}{s^2-9} = \frac{(a+b)s^2 + (c-b)s - 9a - c}{(s-1)(s^2-9)} \longrightarrow \begin{cases} a+b=0, \\ c-b=1, \\ -9a-c=0. \end{cases}$$

   The solution to the above problem is $(\frac{-1}{8}, \frac{1}{8}, \frac{9}{8})$. Therefore,

$$Y(s) = \frac{1}{8}\left(\frac{-1}{s-1} + \frac{s}{s^2-9} + \frac{9}{s^2-9}\right).$$

   Consequently,

$$y(t) = \frac{1}{8}\left(-e^t + \cosh(3t) + 3\sinh(3t)\right).$$

   Note: There is another way to do PFD

$$\frac{s}{(s-1)(s^2-9)} = \frac{a}{s-1} + \frac{b}{s-3} + \frac{c}{s+3}.$$

   You will get $a = \frac{-1}{8}, b = \frac{1}{4}$ and $c = \frac{-1}{8}$. The final answer would be

$$y(t) = \frac{1}{8}\left(-e^t + 2e^{3t} - e^{-3t}\right).$$

5. After applying LT, we get

$$(s^2-2s+5)Y(s) = \frac{s}{s^2+4}+1$$
$$Y(s) = \frac{s}{(s^2-2s+5)(s^2+4)} + \frac{1}{s^2-2s+5}.$$

Let us start with the ILT of the second fraction

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\} = \frac{1}{2}\sin(2t)e^t.$$

To find the ILT of the first fraction, we need to perform a PFD first

$$\frac{s}{(s^2-2s+5)(s^2+4)} = \frac{as+b}{s^2-2s+5} + \frac{cs+d}{s^2+4} = \frac{(a+c)s^3+(b-2c+d)s^2+(4a+5c-2d)s+4b+5c}{(s^2-2s+5)(s^2+4)}$$

This is a 4×4 system but after substituting $a=-c$, we get a 3×3 system:

$$\begin{cases} b-2c+d=0, \\ c-2d=1, \\ 4b+5d=0 \end{cases} \overset{b=2c-d}{\Longrightarrow} \begin{cases} c-2d=1, \\ 8c+d=0 \end{cases} \longrightarrow c=\frac{1}{17},\ d=\frac{-8}{17}.$$

From there, we get $a=\frac{-1}{17}, b=\frac{10}{17}$. Now, we have

$$\frac{s}{(s^2-2s+5)(s^2+4)} = \frac{1}{17}\left(\frac{-s+10}{(s-1)^2+4} + \frac{s-8}{s^2+4}\right),$$
$$= \frac{1}{17}\left(\frac{-(s-1)}{(s-1)^2+4} + \frac{9}{(s-1)^2+4} + \frac{s}{s^2+4} - \frac{8}{s^2+4}\right)$$

The ILT of the expression is

$$\frac{1}{17}\left(-\cos(2t)e^t + \frac{9}{2}\sin(2t)e^t + \cos(2t) - 4\sin(2t)\right).$$

Finally,

$$\boxed{y(t) = \frac{1}{17}\left(-\cos(2t)e^t + \frac{9}{2}\sin(2t)e^t + \cos(2t) - 4\sin(2t)\right) + \frac{1}{2}\sin(2t)e^t}$$

Exercise 15. Apply Laplace transform to this equation and solve it for $Y(s)$

$$y''(t) + y(t) = f(t), \qquad y(0) = 0, y'(0) = 0.$$

Where

$$f(t) = \begin{cases} 1, & 0 \le t \le 2, \\ 0, & t > 2. \end{cases}$$

You do not have to apply the inverse Laplace transform.

Solution 15. The LT of f is

$$F(s) = \int_0^2 e^{-st}\, dt = \frac{1 - e^{-s}}{s}.$$

By applying LT to both sides, we get

$$(s^2 + 1)Y(s) = \frac{1 - e^{-2s}}{s}$$

Therefore,

$$Y(s) = \frac{1 - e^{-2s}}{s(s^2 + 1)}$$

Exercise 16. Use Laplace transform to solve the following IVP

$$y''(t) + 4y(t) = f(t), \qquad y(0) = 0, y'(0) = 0.$$

Where

$$f(t) = \begin{cases} 0, & 0 \le t \le 2, \\ 1, & t > 2. \end{cases}$$

Solution 16. Apply ILT to both sides

$$s^2 Y(s) + 4Y(s) = \frac{1}{s} e^{-2s}.$$

Solve for $Y(s)$

$$Y(s) = \frac{1}{s(s^2+4)} e^{-2s}.$$

Use PFD (details skipped)

$$Y(s) = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) e^{-2s}.$$

Use the second translation theorem to get

$$y(t) = \frac{1}{4}\left(1 - \cos(2(t-2))\right) \mathscr{U}(t-2).$$

Exercise 17. Use Laplace Transform to solve the following IVP:

$$y''(t) - 2y'(t) + y(t) = e^{2t}\mathscr{U}(t-1), \qquad y(0) = 1, y'(0) = 5.$$

Solution 17. First, we need to compute the Laplace transform of the right hand side

$$\mathcal{L}\left\{e^{2t}\mathscr{U}(t-1)\right\} = e^2\mathcal{L}\left\{e^{2(t-1)}\mathscr{U}(t-1)\right\} = e^2e^{-s}\mathcal{L}\left\{e^{2t}\right\} = e^2\frac{e^{-s}}{s-2}.$$

The LT of the left hand side is easy

$$\mathcal{L}\left\{y''(t) - 2y'(t) + y(t)\right\} = (s-1)^2Y(s) - s - 3$$

Therefore

$$Y(s) = \frac{s+3}{(s-1)^2} + e^2\frac{1}{(s-1)^2(s-2)}e^{-s}$$

We have

$$\mathcal{L}^{-1}\left\{\frac{s+3}{(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2} + \frac{4}{(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)} + \frac{4}{(s-1)^2}\right\} = e^t(1+4t)$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2(s-2)}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{s-1} - \frac{1}{(s-1)^2} + \frac{1}{s-2}\right\} = e^{2t} - e^t(t+1).$$

By applying the second translation theorem we get

$$\mathcal{L}^{-1}\left\{e^2\frac{e^{-s}}{(s-1)^2(s-2)}\right\} = e^2\left(e^{2t-2} - e^{t-1}t\right)\mathscr{U}(t-1) = \left(e^{2t} - e^{t+1}t\right)\mathscr{U}(t-1).$$

Combining the two ILT, we get

$$\boxed{y(t) = e^t(1+4t) + \left(e^{2t} - e^{t+1}t\right)\mathscr{U}(t-1)}$$

Exercise 18. Use Laplace Transform to solve the following IVP:

$$y''(t) - 4y'(t) + 5y(t) = e^{3t}\mathscr{U}(t-2), \qquad y(0) = 1, y'(0) = 0.$$

Solution 18. First, we need to compute the Laplace transform of the right hand side

$$\mathcal{L}\left\{e^{3t}\mathscr{U}(t-2)\right\} = e^6\mathcal{L}\left\{e^{3(t-2)}\mathscr{U}(t-1)\right\} = e^6e^{-2s}\mathcal{L}\left\{e^{3t}\right\} = e^6\frac{e^{-2s}}{s-3}.$$

The LT of the left hand side is easy

$$\mathcal{L}\left\{y''(t) - 4y'(t) + 5y(t)\right\} = (s^2 - 4s + 5)Y(s) + s - 4$$

Therefore

$$Y(s) = \frac{s-4}{s^2-4s+5} + e^6\frac{1}{(s^2-4s+5)(s-3)}e^{-2s}$$

We have

$$\mathcal{L}^{-1}\left\{\frac{s-4}{s^2-4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+1} - 2\frac{1}{(s-2)^2+1}\right\} = (\cos(t) - 2\sin(t))\,e^{2t}.$$

Using PFD (details skipped)

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2-4s+5)(s-2)}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-3} - \frac{s-1}{s^2-4s+5}\right\} = \frac{1}{2}e^{3t} - \frac{e^{2t}}{2}\left(\cos(t) + \sin(t)\right).$$

By applying the second translation theorem we get

$$\mathcal{L}^{-1}\left\{e^6\frac{e^{-2s}}{(s^2-4s+5)(s-2)}\right\} = e^6\left(\frac{1}{2}e^{3(t-2)} - \frac{e^{2(t-2)}}{2}\left(\cos(t-2) + \sin(t-2)\right)\right)\mathscr{U}(t-2).$$

Combining the two ILT, we get

$$\boxed{y(t) = (\cos(t) - 2\sin(t))\,e^{2t} + e^6\left(\frac{1}{2}e^{3(t-2)} - \frac{e^{2(t-2)}}{2}\left(\cos(t-2) + \sin(t-2)\right)\right)\mathscr{U}(t-2)}$$

It can be simplified to

$$y(t) = e^{2t}\left[\cos(t) - 2\sin(t) + \frac{1}{2}\left(e^t - e^2\left(\cos(t-2) + \sin(t-2)\right)\right)\mathscr{U}(t-2)\right]$$

Exercise 19. Use theorem 7.4.1 to compute $\mathcal{L}\left\{t^3\cos(t)\right\}$.

Solution 19. We have

$$\mathcal{L}\left\{t^3\cos(t)\right\} = -\frac{\mathrm{d}^3}{\mathrm{d}s^3}\mathcal{L}\left\{\cos(t)\right\} = -\frac{\mathrm{d}^3}{\mathrm{d}s^3}\left(\frac{s}{s^2+1}\right)$$

$$= -\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{1-s^2}{\left(s^2+1\right)^2}\right)$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{2s\left(s^2-3\right)}{\left(s^2+1\right)^3}\right)$$

$$= -\left(-\frac{6\left(s^4-6s^2+1\right)}{\left(s^2+1\right)^4}\right)$$

$$= -\frac{6\left(s^4-6s^2+1\right)}{\left(s^2+1\right)^4}$$

There are other equivalent forms that have $(s^2+1)^8$ in the denominator.

Exercise 20. Write the following inverse Laplace transforms as convolutions (You do not need to evaluate the integral):

(a) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-4)s}\right\}$,

(b) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$.

(c) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2-2s+2)(s^2+2s+2)}\right\}$.

(d) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+4)(s^2+1)}\right\}$.

Solution 20. (a)

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-4)s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} * \left\{\frac{1}{s-4}\right\} = (1) * (e^{4t}) = \int_0^t e^{4r}\, dr$$

(b)

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)}\right\}$$

If $a \neq 0$, we have

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)}\right\} = \frac{1}{a}(\cos(at)) * (\sin(at)) = \frac{1}{a}\int_0^t \cos(ar)\sin(a(t-r))\, dr$$

If $a = 0$, then

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2}.$$

(c)

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s^2-2s+2)(s^2+2s+2)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} \\ &= (e^t\sin(t)) * (e^{-t}\sin(t)) \\ &= \int_0^t e^{r-(t-r)}\sin(r)\sin(t-r)\, dr \\ &= \int_0^t e^{2r-t}\sin(r)\sin(t-r)\, dr\end{aligned}$$

(d) This one is very similar to (b):

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = (\cos(2t)) * (\sin(t)) = \int_0^t \cos(2r)\sin(t-$$

Exercise 21. Find the following Laplace transforms without evaluating the integral first

(a) $\mathcal{L}\left\{\int_0^t e^r\,dr\right\}$

(b) $\mathcal{L}\left\{\int_0^t re^r\,dr\right\}$

(c) $\mathcal{L}\left\{t\int_0^t re^r\,dr\right\}$

(d) $\mathcal{L}\left\{\int_0^t r\sin(r)\,dr\right\}$

Solution 21. (a) Apply theorem 7.4.3 to get

$$\mathcal{L}\left\{\int_0^t e^r\,dr\right\} = \frac{1}{s}\mathcal{L}\left\{e^t\right\} = \frac{1}{s(s-1)}.$$

(b) This is similar to (a),

$$\mathcal{L}\left\{\int_0^t re^r\,dr\right\} = \frac{1}{s}\mathcal{L}\left\{te^t\right\} = \frac{1}{s(s-1)^2}.$$

The first translation theorem can be applied to get $\mathcal{L}\left\{te^t\right\} = \frac{1}{(s-1)^2}$.

(c) Use the derivative of transforms theorem to get

$$\mathcal{L}\left\{t\int_0^t re^r\,dr\right\} = -\frac{d}{ds}\mathcal{L}\left\{\int_0^t re^r\,dr\right\} = -\frac{d}{ds}\frac{1}{(s-1)^2} = \frac{3s-1}{s^2(s-1)^3}.$$

(d)

$$\mathcal{L}\left\{\int_0^t r\sin(r)\,dr\right\} = \frac{1}{s}\mathcal{L}\left\{t\sin(t)\right\} = \frac{-1}{s}\frac{d}{ds}\frac{1}{s^2+1} = \frac{2}{\left(s^2+1\right)^2}.$$

Exercise 22. Use Laplace transform to the following (so called integro-differential equation) to find $y(t)$

$$y'(t) = 1 - \sin(t) - \int_0^t y(r)\, dr, \qquad y(0) = 0.$$

Note: The steps are similar as for ODES: Apply LT, solve for $Y(s)$, apply ILT.

Solution 22. Apply LT to both sides to get

$$sY(s) = \frac{1}{s} - \frac{1}{s^2+1} - \frac{1}{s}Y(s)$$

This can be simplified to

$$(s^2+1)Y(s) = 1 - \frac{s}{s^2+1}$$

so

$$Y(s) = \frac{1}{s^2+1} - \frac{s}{(s^2+1)^2}$$

To compute the LT of the second part, we use the convolution theorem

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\cdot\frac{1}{s^2+1}\right\} = \int_0^t \sin(r)\cos(t-r)\, dr$$

To compute this integral we can use $\sin(a)\cos(b) = \frac{1}{2}\left(\sin(a+b) + \sin(a-b)\right)$

$$\begin{aligned}
\int_0^t \sin(r)\cos(t-r)\, dr &= \frac{1}{2}\int_0^t \sin(t)\, dr + \frac{1}{2}\int_0^t \sin(2r-t)\, dr \\
&= \frac{1}{2}t\sin(t) + \frac{1}{2}\left[\frac{-\cos(2r-t)}{2}\right]_{r=0}^{t} \\
&= \frac{1}{2}t\sin(t) + 0 \\
&= \frac{1}{2}t\sin(t).
\end{aligned}$$

The final answer is

$$\boxed{y(t) = \sin(t) - \frac{1}{2}t\sin(t).}$$

Exercise 23. f is a $2\pi-$periodic function, defined on $[0, 2\pi)$ by

$$f(t) = \begin{cases} \sin(t), & 0 \le t < \pi, \\ 0, & \pi \le t < 2\pi \end{cases}$$

Here's the graph of f:

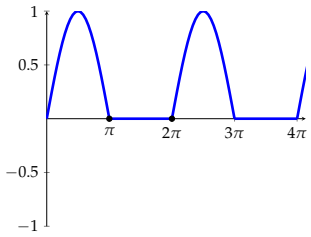


Figure 3: The graph of f

Find the Laplace transform of f.

Solution 23. Using the theorem about periodic functions, we have

$$F(s) = \frac{1}{1 - e^{-2\pi s}} \underbrace{\int_0^\pi \sin(t) e^{-st}\, dt}_{\text{Let us call this } I}$$

There are at least two ways to compute I:

1. 1st method: Integrate by parts twice and solve for I

$$\begin{aligned} I &= \int_0^\pi \sin(t) e^{-st}\, dt \\ &= \Big[-\cos(t) e^{-st} \Big]_{0^\pi} - s \int_0^\pi \cos(t) e^{-st}\, dt \\ &= e^{-\pi s} + 1 - s \int_0^\pi \cos(t) e^{-st}\, dt \\ &= e^{-\pi s} + 1 - s \Big[\sin(t) e^{-st} \Big]_0^\pi - s^2 \int_0^\pi \sin(t) e^{-st}\, dt \\ I &= e^{-\pi s} + 1 - s^2 I. \end{aligned}$$

And then you can solve for I to get $I = \dfrac{e^{-\pi s} + 1}{s^2 + 1}$. Finally we have

$$F(s) = \frac{1}{1 - e^{-2\pi s}} \cdot \frac{e^{-\pi s} + 1}{s^2 + 1}.$$

This can be simplified a little bit but there is no need for that.

2. 2nd method: we can write I as

$$\begin{aligned} I &= \int_0^\infty \sin(t)\left(1-\mathscr{U}\left(t-\pi\right)\right)e^{-st}\,dt \\ &= \mathcal{L}\left\{\sin(t)\right\} - \mathcal{L}\left\{\sin(t)\mathscr{U}\left(t-\pi\right)\right\} \\ &= \mathcal{L}\left\{\sin(t)\right\} + \mathcal{L}\left\{\sin(t-\pi)\mathscr{U}\left(t-\pi\right)\right\} \\ &= \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}. \end{aligned}$$

As you can see, no integration skills were needed here. This shows the importance of algebraic manipulations and the basic theorems of Laplace transform.

Note: There are other methods to evaluate I. For example, if you are comfortable with complex numbers, you can evaluate

$$\int_0^\pi e^{it-st}\,dt$$

and take the imaginary part.

Exercise 24. f is a $2-$periodic function, defined on $[0,2)$ by

$$f(t) = \begin{cases} 1, & 0 \le t < 1, \\ \frac{1}{2}, & 1 \le t < 2 \end{cases}$$

Here's the graph of f:

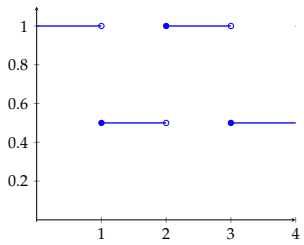


Figure 4: The graph of f

Find the Laplace transform of f.

Solution 24. Using the theorem, we have

$$F(s) = \frac{1}{1-e^{-2s}} \int_0^2 f(t)e^{-st}\,dt$$

Because the function f is defined piecewise, we need to break the integral down into two:

$$\begin{aligned} \int_0^2 f(t)e^{-st}\,dt &= \int_0^1 e^{-st}\,dt + \frac{1}{2}\int_1^2 e^{-st}\,dt \\ &= \frac{1-e^{-s}}{s} + \frac{e^{-s}-e^{-2s}}{2s} \\ &= \frac{1}{s} - \frac{e^{-s}}{2s} - \frac{e^{-2s}}{2s} \end{aligned}$$

Therefore,

$$\boxed{F(s) = \frac{1}{1-e^{-2s}}\left(\frac{1}{s} - \frac{e^{-s}}{2s} - \frac{e^{-2s}}{2s}\right)}$$

Exercise 25. Use Laplace transform to solve the following system:

$$\begin{cases} x'(t) + y'(t) = -3x(t) + 2 \\ x'(t) - 2y'(t) = -8t \end{cases} \qquad x(0) = 1, \quad y(0) = -1.$$

Solution 25. Apply Laplace transform to both sides to get

$$\begin{bmatrix} s+3 & s \\ s & -2s \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} \frac{2}{s} \\ \frac{-8}{s^2} + 3 \end{bmatrix}.$$

Multiply both sides with the inverse matrix to get

$$\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \frac{-1}{3s(s+2)} \begin{bmatrix} -2s & -s \\ -s & s+3 \end{bmatrix} \begin{bmatrix} \frac{2}{s} \\ \frac{-8}{s^2} + 3 \end{bmatrix} = \frac{-1}{3s(s+2)} \begin{bmatrix} -3s - 4 + \frac{8}{s} \\ 3s - \frac{8}{s} - \frac{24}{s^2} + 7 \end{bmatrix}.$$

Which leads to

$$\begin{cases} X(s) = \dfrac{1}{s+2} + \dfrac{4}{3s(s+2)} - \dfrac{8}{3s^2(s+2)}, \\ \\ Y(s) = \dfrac{-1}{s+2} + \dfrac{8}{3s^2(s+2)} + \dfrac{8}{s^3(s+2)} - \dfrac{7}{3s(s+2)}. \end{cases}$$

We notice that there are three inverse LTs to be computed, they can be computed using the convolution theorem easily: We know that $\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$, then

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\right\} = \int_0^t e^{-2r}\, dr = \frac{1 - e^{-2t}}{2}$$

Again,

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+2)}\right\} = \int_0^t \frac{1 - e^{-2r}}{2} dr = \frac{t}{2} - \frac{1 - e^{-2t}}{4}$$

Again,

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s+2)}\right\} = \int_0^t \frac{r}{2} - \frac{1 - e^{-2r}}{4} dr = \frac{t^2}{4} - \frac{t}{4} + \frac{1 - e^{-2t}}{8}$$

By putting everything together, you should get

$$\begin{cases} x(t) = \dfrac{4}{3} - \dfrac{4t}{3} - \dfrac{e^{-2t}}{3}, \\ \\ y(t) = 2t^2 - \dfrac{2t}{3} - \dfrac{e^{-2t}}{6} - \dfrac{5}{6}. \end{cases}$$

Exercise 26. Use Laplace transform to solve the following initial value problem:

$$y''(t) + y(t) = \delta(t - 2\pi) + \delta(t - 4\pi), \qquad y(0) = 1,\ y'(0) = 0$$

Solution 26. Apply LT to both sides and solve the equation for $Y(s)$ to get

$$Y(s) = \frac{s}{s^2 + 1} + \frac{e^{-2\pi s}}{s^2 + 1} + \frac{e^{-4\pi s}}{s^2 + 1}.$$

Apply the inverse LT to both sides (use the second translation theorem) to get

$$y(t) = \cos(t) + \sin(t - 2\pi)\mathscr{U}(t - 2\pi) + \sin(t - 4\pi)\mathscr{U}(t - 4\pi).$$

Note: The above expression is equivalent to

$$y(t) = \cos(t) + \sin(t)\mathscr{U}(t - 2\pi) + \sin(t)\mathscr{U}(t - 4\pi).$$

Exercise 27. Let a be a positive number, solve the following IVP

$$y'(t) = \delta(t-a), \qquad y(0) = 0.$$

Note: This problem shows that the delta "function" is the "derivative" of another function that we have studied this semester.

Solution 27. Apply LT to both sides

$$sY(s) = e^{-as}$$

divide by s

$$Y(s) = \frac{1}{s}e^{-as}$$

Apply the second translation theorem to get

$$y(t) = \mathscr{U}(t-a).$$

Exercise 28. In lesson 12, we have seen that the set

$$S = \{p_0, p_1, p_2\}, \qquad p_0(x) = 1,\ p_1(x) = x,\ p_2(x) = \frac{1}{2}\left(1 - 3x^2\right),$$

is an orthogonal set.

1. Find the norms of p_0, p_1 and p_2 on $[-1, 1]$.
2. Let $f(x) = e^x$. Find the following inner products on $[-1, 1]$:

$$(p_0, f), \quad (p_1, f), \quad (p_2, f),$$

3. Find the orthogonal projection of f onto the span of S. That is,

$$\sum_{i=0}^{2} \frac{(p_i, f)}{\|p_i\|^2} p_i(x) = \frac{(p_0, f)}{\|p_0\|^2} p_0(x) + \frac{(p_1, f)}{\|p_1\|^2} p_1(x) + \frac{(p_2, f)}{\|p_2\|^2} p_2(x)$$

Note: Notice that the term $\frac{(p_i, f)}{\|p_i\|^2}$ for $i = 0, 1, 2$ is a real number and can be obtained from the first two questions. Therefore, the projection will be a second order polynomial. If you plot the projection and the function f on $[-1, 1]$, their graphs should be close to each other.

Solution 28. 1. We square and integrate p_0, p_1, p_2, then we take the square root:

(a)

$$\int_{-1}^{1} p_0(x)^2 \, dx = \int_{-1}^{1} 1 \, dx = 2.$$

Therefore, $\|p_0\| = \sqrt{2}$.

(b)

$$\int_{-1}^{1} p_1(x)^2 \, dx = \int_{-1}^{1} x^2 \, dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}.$$

Therefore, $\|p_1\| = \sqrt{\frac{2}{3}}$.

(c)

$$\int_{-1}^{1} p_2(x)^2 \, dx = \frac{1}{4}\int_{-1}^{1} (1 - 3x^2)^2 \, dx = \frac{1}{4}\int_{-1}^{1} 1 + 9x^4 - 6x^2 \, dx = \frac{1}{4}\left[x - \frac{9}{5}x^5 - 2x^3\right]_{-1}^{1} = \frac{2}{5}.$$

Therefore, $\|p_2\| = \sqrt{\frac{2}{5}}$.

2. As you will see, when computing (p_1, f), we can use what we have found for (p_0, f). We can do a similar thing for (p_2, f). This will make our job much easier.

(a)

$$(p_0, f) = \int_{-1}^{1} e^x \, dx = e - e^{-1}.$$

(b) Use IBP:

$$(p_1, f) = \int_{-1}^{1} xe^x \, dx = \Big[xe^x\Big]_{-1}^{1} - \int_{-1}^{1} e^x \, dx$$

The last integral was already computed, so

$$(p_1, f) = e + e^{-1} - (e - e^{-1}) = 2e^{-1}.$$

(c) Same thing, use IBP

$$(p_2, f) = \frac{1}{2}\int_{-1}^{1} (1 - 3x^2)e^x \, dx = \frac{1}{2}\Big[(1 - 3x^2)e^x\Big]_{-1}^{1} + 3\int_{-1}^{1} xe^x \, dx$$

The last integral was already computed and it is equal to $2e^{-1}$, so

$$(p_2, f) = \frac{1}{2}\Big[-2e + 2e^{-1}\Big] + 6e^{-1} = 7e^{-1} - e.$$

3. We just need to put everything together: The orthogonal projection of f onto the span of S is

$$\boxed{\frac{e - e^{-1}}{2} + 3e^{-1}x + \frac{5}{4}\left(7e^{-1} - e\right)(1 - 3x^2).}$$

Exercise 29. Find a non-zero quadratic polynomial that is orthogonal to x, x^2 and x^3 on the interval $[-1, 1]$.

Hint: start by assuming that the polynomial that we are seeking is $p(x) = ax^2 + bx + c$ and find a, b, c.

Solution 29. We want to find a, b, c such that

$$\int_{-1}^{1} x(ax^2 + bx + c)\, dx = 0, \qquad \int_{-1}^{1} x^2(ax^2 + bx + c)\, dx = 0, \qquad \int_{-1}^{1} x^3(ax^2 + bx + c)\, dx = 0.$$

After computing the integrals, we get the following system

$$\begin{cases} \dfrac{2b}{3} = 0, \\ \dfrac{2a}{5} + \dfrac{2c}{3} = 0, \\ \dfrac{2b}{5} = 0. \end{cases}$$

Which has an infinite number of solutions $b = 0$ and $5c = -3a$. For example: $p(x) = -5x^2 + 3$.

Note: any scalar multiple of $-5x^2 + 3$ is a solution as well.

Exercise 30. Compute the following integral

$$\int_0^\pi \sin(nx)|\sin(2x)|\,dx,$$

where n is a natural number. You may want to follow the following hints:

(a) Decompose the integral into two integrals to get rid of the absolute value.

(b) Use the identity $\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$ to simplify the integrand.

(c) Discuss the case $n = 2$ separately to avoid divisions by zero.

Note: This is coefficient of expansion of $|\sin(2x)|$ in the basis $\{\sin(nx)\}_{n=1}^\infty$ before dividing by the norm.

Solution 30. Let us call this integral b_n, we have

$$\begin{aligned}
b_n &= \int_0^\pi \sin(nx)|\sin(2x)|\,dx \\
&= \int_0^{\frac{\pi}{2}} \sin(nx)\sin(2x)\,dx - \int_{\frac{\pi}{2}}^\pi \sin(nx)\sin(2x)\,dx \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos((n-2)x) - \cos((n+2)x)\,dx - \frac{1}{2}\int_{\frac{\pi}{2}}^\pi \cos((n-2)x) - \cos((n+2)x)\,dx \\
&= \frac{1}{2}\left[\frac{\sin((n-2)x)}{n-2} - \frac{\sin((n+2)x)}{n+2}\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\left[\frac{\sin((n-2)x)}{n-2} - \frac{\sin((n+2)x)}{n+2}\right]_{\frac{\pi}{2}}^{\pi} \\
&= \frac{\sin((n-2)\frac{\pi}{2})}{n-2} - \frac{\sin((n+2)\frac{\pi}{2})}{n+2} \\
&= \frac{\sin(\frac{n\pi}{2} - \pi)}{n-2} - \frac{\sin(\frac{n\pi}{2}+\pi)}{n+2}.
\end{aligned}$$

This works well when $n \neq 2$. However, when $n = 2$, we will have an issue going from the third line to the fourth line, in that case, we have

$$\begin{aligned}
b_2 &= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 - \cos(4x)\,dx - \frac{1}{2}\int_{\frac{\pi}{2}}^\pi 1 - \cos(4x)\,dx \\
&= \frac{1}{2}\left[x - \frac{\sin(4x)}{4}\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\left[x - \frac{\sin(4x)}{4}\right]_{\frac{\pi}{2}}^{\pi} \\
&= 0.
\end{aligned}$$

Note: We can simplify b_n further using $\sin(x+\pi) = \sin(x-\pi) = -\sin(x)$:

$$b_n = \sin\left(\frac{n\pi}{2}\right)\left(\frac{1}{n+2} - \frac{1}{n-2}\right) = \frac{-4\sin\left(\frac{n\pi}{2}\right)}{n^2-4}.$$

Exercise 31. Redo the previous exercise with $\cos(nx)$ instead. That is, compute

$$\int_0^{\pi} \cos(nx)|\sin(2x)|\,dx$$

Solution 31. We proceed the same way we did in exercise two, this time we use

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

Let us the given integral a_n:

$$\begin{aligned}
a_n &= \int_0^{\pi} \cos(nx)|\sin(2x)|\,dx \\
&= \int_0^{\frac{\pi}{2}} \cos(nx)\sin(2x)\,dx - \int_{\frac{\pi}{2}}^{\pi} \cos(nx)\sin(2x)\,dx \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} -\sin((n-2)x) + \sin((n+2)x)\,dx - \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} -\sin((n-2)x) + \sin((n+2)x)\,dx \\
&= \frac{1}{2}\left[\frac{\cos((n-2)x)}{n-2} - \frac{\cos((n+2)x)}{n+2}\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\left[\frac{\cos((n-2)x)}{n-2} - \frac{\cos((n+2)x)}{n+2}\right]_{\frac{\pi}{2}}^{\pi} \\
&= \frac{1}{2}\left(\frac{\cos((n-2)\frac{\pi}{2})-1}{n-2} - \frac{\cos((n+2)\frac{\pi}{2})-1}{n+2}\right) \\
&\quad -\frac{1}{2}\left(\frac{\cos((n-2)\pi) - \cos((n-2)\frac{\pi}{2})}{n-2} - \frac{\cos((n+2)\pi) - \cos((n+2)\frac{\pi}{2})}{n+2}\right)
\end{aligned}$$

When $n = 2$, we can go back to the third line and pick up from there

$$a_2 = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(4x)dx - \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \sin(4x)dx = \frac{1}{2}\left[\frac{-\cos(4x)}{4}\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\left[\frac{-\cos(4x)}{4}\right]_{\frac{\pi}{2}}^{\pi} = 0.$$

Note: We can simplify a_n greatly, we have

$$\cos((n-2)\pi) = \cos((n+2)\pi) = (-1)^n, \quad \text{and} \quad \cos((n-2)\frac{\pi}{2}) = \cos((n+2)\frac{\pi}{2}) = -\cos(\frac{n\pi}{2}),$$

then

$$\boxed{a_n = -\frac{4\cos(\frac{n\pi}{2}) + 2 + 2(-1)^n}{n^2-4}}$$

Exercise 32. Find the Fourier series of $f(x) = |\sin(x)|$ on $(-\pi, \pi)$. Hint: Be careful.

Solution 32. f is even, then $b_n = 0$ for any integer $n \geq 1$. We only need to find a_0 and a_n for $n \geq 1$. Let us start with a_0:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin(x)|\, dx = \frac{2}{\pi}\int_0^{\pi} \sin(x)\, dx = \frac{2}{\pi}\Big[-\cos(x)\Big]_0^{\pi} = \frac{4}{\pi}.$$

Now let us move to a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin(x)|\cos(nx)\, dx = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\, dx$$

Now, use the product-to-sum formula to get

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin((n+1)x) - \sin((n-1)x)dx = \frac{1}{\pi}\left(\int_0^{\pi} \sin((n+1)x)\, dx - \int_0^{\pi} \sin((n-1)x)dx\right)$$

The first integral is pretty straightforward

$$\int_0^{\pi} \sin((n+1)x)\, dx = \frac{1+\cos(n\pi)}{n+1} = \frac{1+(-1)^n}{n+1}$$

The second integral should be dealt with caution, if $n = 1$ then $\int_0^{\pi} \sin((n-1)x)dx = 0$. If $n > 1$, then

$$\int_0^{\pi} \sin((n-1)x)dx = \frac{1+\cos(n\pi)}{n-1} = \frac{1+(-1)^n}{n-1}$$

Which means that

$$a_1 = \frac{1+(-1)^1}{1+1} - 0 = 0.$$

And for $n > 1$, we have

$$a_n = \frac{1+(-1)^n}{n+1} - \frac{1+(-1)^n}{n-1} = \frac{-2(1+(-1)^n)}{n^2-1}$$

Therefore, the Fourier series of f is

$$\boxed{\frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{1+(-1)^n}{n^2-1}\cos(nx)}$$

Note 1: Notice that wee had to discuss the case $n = 1$ separately to avoid having a division by zero. (See the hint).

Note 2: More simplifications can be applied, for example, the summand is zero when n is odd, so the Fourier series can be written as

$$\frac{2}{\pi} - \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{1+(-1)^{2k}}{(2k)^2-1}\cos(2kx) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2kx)}{4k^2-1}.$$

Exercise 33. Compute the Fourier series of the following functions on $(-\pi, \pi)$, you should simplify the coefficients.

(a) x^2.

(b) e^{2x}.

(c) $\sinh(x)$.

(d) $\cos(2x)^2$.

Note: The last one is not as friendly as it might seem. Think about possible divisions by zero.

Solution 33. (a) Notice that $b_n = 0$ (because $x \mapsto x^2$ is even). We need to find a_0 and a_n. We will employ the parity to reduce the interval of integration to $(0, \pi)$

$$a_0 = \frac{2}{\pi}\int_0^\pi x^2\,dx = \frac{2\pi^2}{3}.$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi x^2\cos(nx)\,dx \\
&= \underbrace{\frac{2}{\pi n}\Big[x^2\sin(nx)\Big]_{x=0}^{x=\pi}}_{=0} - \frac{4}{\pi n}\int_0^\pi x\sin(nx)\,dx \\
&= \frac{4}{\pi n^2}\Big[x\cos(nx)\Big]_{x=0}^{x=\pi} - \frac{4}{\pi n^2}\underbrace{\int_0^\pi \cos(nx)\,dx}_{=0}, \\
&= \frac{4\cos(n\pi)}{n^2}, \\
&= \frac{4(-1)^n}{n^2}
\end{aligned}$$

Finally, we have

$$\boxed{S(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx).}$$

(b) There are multiple ways to approach. The brute-force approach would be computing a_n and b_n separately by integrating by parts twice each time. Here's a better (shorter) approach:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{2x}\cos(nx)\,dx = \underbrace{\frac{1}{\pi n}\Big[e^{2x}\sin(nx)\Big]_{x=-\pi}^{\pi}}_{=0} - \frac{2}{\pi n}\int_{-\pi}^{\pi} e^{2x}\sin(nx)\,dx = \frac{-2}{n}b_n.$$

So, we have $a_n = \frac{-1}{n}b_n$. Also, we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{2x}\sin(nx)\,dx = \frac{1}{\pi n}\Big[-e^{2x}\cos(nx)\Big]_{x=-\pi}^{\pi} + \frac{2}{\pi n}\int_{-\pi}^{\pi} e^{2x}\cos(nx)\,dx$$

$$= \frac{(-1)^{n+1}(e^{2\pi}-e^{-2\pi})}{n\pi} + \frac{2}{n}a_n$$

By replacing a_n by $\frac{-2}{n}b_n$ and solving the equation, we get

$$b_n = \frac{n(-1)^{n+1}(e^{2\pi}-e^{-2\pi})}{\pi(n^2+4)}.$$

Consequently, we have

$$a_n = \frac{2(-1)^{n}(e^{2\pi}-e^{-2\pi})}{\pi(n^2+4)}.$$

Now, a_0 can be computed with ease

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{2x}\,dx = \frac{e^{2\pi}-e^{-2\pi}}{2\pi}.$$

Finally, we have

$$\boxed{S(x) = \frac{e^{2\pi}-e^{-2\pi}}{4\pi} + \sum_{n=1}^{\infty}\frac{2(-1)^{n}(e^{2\pi}-e^{-2\pi})}{\pi(n^2+4)}\cos(nx) + \frac{n(-1)^{n+1}(e^{2\pi}-e^{-2\pi})}{\pi(n^2+4)}\sin(nx)}$$

Another (even shorter) approach: Notice that

$$a_n + ib_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{2x}e^{inx}\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{(2+in)x}\,dx = \frac{e^{2\pi+n\pi i}-e^{-2\pi-n\pi i}}{\pi(2+ni)} = \frac{(-1)^{n}(e^{2\pi}-e^{-2\pi})}{\pi(2+ni)}$$

To get the real an imaginary part, we multiply and divide by $2-ni$ (the conjugate of the denominator):

$$a_n + ib_n = \frac{(-1)^n(e^{2\pi} - e^{-2\pi})(2 - ni)}{\pi(4 + n^2)} = \underbrace{\frac{2(-1)^n(e^{2\pi} - e^{-2\pi})}{\pi(4 + n^2)}}_{=a_n} + \left(\underbrace{\frac{(-1)^{n+1}(e^{2\pi} - e^{-2\pi})n}{\pi(4 + n^2)}}_{=b_n} \right) i$$

(c) This function is odd, so $a_0 = a_n = 0$, we just need to find b_n:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sinh(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} \sinh(x)\sin(nx)\,dx.$$

We perform IBP twice to get

$$b_n = \frac{2}{\pi}\Big[\cosh(x)\sin(nx)\Big]_0^{\pi} - \frac{2n}{\pi}\int_0^{\pi}\cosh(x)\cos(nx)\,dx \qquad \text{(IBP)}$$

$$= \frac{-2n}{\pi}\int_0^{\pi}\cosh(x)\cos(nx)\,dx$$

$$= \frac{-2n}{\pi}\Big[\sinh(x)\cos(nx)\Big]_0^{\pi} - \frac{2n^2}{\pi}\int_0^{\pi}\sinh(x)\sin(nx)\,dx$$

$$b_n = \frac{-2n(-1)^n\sinh(\pi)}{\pi} - n^2 b_n$$

We solve the last equation for b_n to get

$$b_n = \frac{-2n(-1)^n\sinh(\pi)}{\pi(n^2+1)}.$$

Therefore, the Fourier series of f is

$$S(x) = \sum_{n=1}^{\infty}\frac{-2n(-1)^n\sinh(\pi)}{\pi(n^2+1)}\sin(nx).$$

(d) Since f is even, we only need to compute a_0, a_n. First,we can linearize f:

$$f(x) = \cos(2x)^2 = \frac{1}{2} + \frac{1}{2}\cos(4x).$$

Let us start with a_0:

$$a_0 = \frac{2}{\pi}\int_0^\pi \frac{1}{2} + \frac{1}{2}\cos(4x)dx = \frac{1}{\pi}\left[x + \frac{\sin(4x)}{4}\right]_0^\pi = 1.$$

Next, we will use the following identity to compute a_n

$$\boxed{\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]}.$$

We break a_n into three integrals

$$a_n = \frac{1}{\pi}\int_0^\pi \cos(nx)dx + \frac{1}{2\pi}\int_0^\pi \cos((n+4)x)dx + \frac{1}{2\pi}\int_0^\pi \cos((n-4)x)dx$$

The first integral is the easiest

$$\frac{1}{\pi}\int_0^\pi \cos(nx)dx = \frac{1}{\pi}\left[\frac{\sin(nx)}{n}\right]_0^\pi = 0.$$

The second one can be done similarly:

$$\frac{1}{2\pi}\int_0^\pi \cos((n+4)x)dx = \frac{1}{2\pi}\left[\frac{\sin((n+4)x)}{n+4}\right]_0^\pi = 0.$$

However, when we try to do the same thing for the last integral we see that there is a possible division by zero

$$\frac{1}{2\pi}\int_0^\pi \cos((n-4)x)dx = \frac{1}{2\pi}\left[\frac{\sin((n-4)x)}{n-4}\right]_0^\pi = 0, \qquad \text{works for } n \neq 4.$$

Hence, we have this integral for $n = 4$ separately, it turns out to be a simple integral

$$\frac{1}{2\pi}\int_0^\pi \cos((4-4)x)dx = \frac{1}{2\pi}\int_0^\pi 1dx = \frac{1}{2}.$$

Finally, we have

$$a_0 = 1, \quad a_4 = \frac{1}{2}, \quad a_n = 0 \text{ for any } n \neq 4 \text{ and } n > 0,$$

This means that $a_0 = 1$ and a_n is zero for every n except 4. Therefore,

$$S(x) = \frac{a_0}{2} + a_4\cos(4x) = \frac{1}{2} + \frac{1}{2}\cos(4x).$$

Note: Notice that the Fourier series is equal to the original function.

Exercise 34. In this exercise, we want to compute the Fourier coefficients of e^{-x} on $(-\pi, \pi)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\cos(nx)dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\sin(nx)dx$$

However, you might have noticed that the IBP approach, although effective, was a bit tedious. Here's a new approach:

$$\begin{aligned}
e^{inx} &= \cos(nx) + i\sin(nx) && \text{(Euler's identity)}\\
e^{-x}e^{inx} &= e^{-x}\cos(nx) + ie^{-x}\sin(nx) && \text{(Multiply by } e^{-x}\text{)}\\
\int_{-\pi}^{\pi} e^{-x}e^{inx}dx &= \int_{0}^{\pi} e^{-x}\cos(nx)dx + i\int_{-\pi}^{\pi}\sin(nx)dx && \text{(Integrate)}\\
\frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}e^{inx}dx &= a_n + ib_n && (1)
\end{aligned}$$

Now, we got rid of trig functions and we can proceed

(a) Compute the integral in equation (1). That would give you $a_n + ib_n$.

(b) Write the result from the previous question as

$$\Big(\dots\Big) + i\Big(\dots\Big)$$

where the expressions in each bracket are real.

Hint: You might need to multiply by a conjugate somewhere.

(c) By mere observation, what are a_n and b_n?

Solution 34. (a) We have

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}e^{inx}dx = \frac{1}{\pi}\int_{-\pi}^{p} ie^{(in-1)x} &= \frac{1}{\pi}\left[\frac{e^{(in-1)x}}{in-1}\right]_{-\pi}^{\pi}\\
&= \frac{e^{(in-1)\pi} - e^{-(in-1)\pi}}{\pi(in-1)}
\end{aligned}$$

(b) To simplify the previous expression, we first use Euler's identity to get $e^{in\pi} = e^{-in\pi} = \cos(n\pi) = (-1)^n$. Now, we have

$$\frac{e^{(in-1)\pi} - e^{-(in-1)\pi}}{\pi(in-1)} = \frac{(-1)^n\left(e^{\pi} - e^{-\pi}\right)}{\pi(in-1)}$$

Now, we multiply the top and the bottom by the conjugate $in+1$ to get

$$\begin{aligned}
\frac{(-1)^n\left(e^{\pi} - e^{-\pi}\right)}{\pi(in-1)}\cdot\frac{in+1}{in+1} &= \frac{(-1)^n\left(e^{\pi} - e^{-\pi}\right)(in+1)}{\pi(n^2+1)}\\
&= \frac{(-1)^n\left(e^{\pi} - e^{-\pi}\right)}{\pi(n^2+1)} + i\left(\frac{(-1)^n\left(e^{\pi} - e^{-\pi}\right)n}{\pi(n^2+1)}\right)
\end{aligned}$$

(c) From the previous result, we have

$$a_n + ib_n = \frac{(-1)^n \left(e^{\pi} - e^{-\pi}\right)}{\pi(n^2+1)} + i\left(\frac{(-1)^n \left(e^{\pi} - e^{-\pi}\right) n}{\pi(n^2+1)}\right)$$

This implies that

$$a_n = \frac{(-1)^n \left(e^{\pi} - e^{-\pi}\right)}{\pi(n^2+1)}, \qquad b_n = \frac{(-1)^n \left(e^{\pi} - e^{-\pi}\right) n}{\pi(n^2+1)}$$

Exercise 35. Let f be defined on $(-\pi, \pi)$ by

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x^2, & 0 < x < \pi. \end{cases}$$

(a) Compute and simplify $S(x)$ (the Fourier series of f).

(b) Simplify the expression of $S(\pi)$. (It will involve a sum).

(c) Using Dirichlet's theorem, find $S(\pi)$. (Hint: think about the periodic extension of f, what is $f(\pi^-)$ and $f(\pi^+)$).

(d) Use (b) and (c) to show that

$$\boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots}$$

Solution 35. 1. I will skip the details here:

$$S(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} \cos(nx) + \left(\frac{(-1)^{n+1}\pi}{n} + \frac{2((-1)^n - 1)}{\pi n^3} \right) \sin(nx)$$

2. We have $\cos(n\pi) = (-1)^n$ and $\sin(n\pi) = 0$. Therefore

$$S(\pi) = \frac{\pi^2}{6} + 2 \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

3. If we extend f periodically, we get $f(\pi^+) = 0$ and $f(\pi^-) = \pi^2$. Therefore

$$S(\pi) = \frac{\pi^2}{2}.$$

4. From (b) and (c), we have

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

divide by 2 to get

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

subtract $\frac{\pi^2}{12}$ from both sides to get

$$\boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}}$$

Exercise 36. Let

$$f(x) = \begin{cases} 0, & -2 < x < 0, \\ 1, & 0 \leq x < 2, \end{cases}$$

and let $S(x)$ be its Fourier series. Using Dirichlet's theorem, what are the values of $S(0), S(1), S(-2)$, and $S(2)$?

Solution 36.

1. We have $f(0^+) = 1$ and $f(0^-) = 0$. Therefore, $S(0) = \frac{1}{2}$.
2. The function f is continuous at 1, so $S(1) = f(1) = 1$.

   Another way to do it would be by taking the average $\frac{f(1^-)+f(1^+)}{2} = \frac{1+1}{2}$.
3. From the definition of f, we have $f(2^-) = 1$. If we extend f periodically, we get $f(2^+) = 0$. Therefore, $S(2) = \frac{1}{2}$.
4. A similar argument to (3) gives us $S(-2) = \frac{1}{2}$.

Exercise 37. Let $f(x) = x\sin(x)$.

1. Find the cosine series of f on $(0, \pi)$.
2. Find the sine series of f on $(0, \pi)$.

Note: It goes without saying that "you should not divide by zero". See my comment on the third problem.

Solution 37. (a)

$$a_0 = \frac{2}{\pi}\int_0^\pi x\sin(x)\,dx = \frac{2}{\pi}\left(\Big[-x\cos(x)\Big]_0^\pi + \underbrace{\int_0^\pi \cos(x)\,dx}_{=0}\right) = 2$$

$$a_n = \frac{2}{\pi}\int_0^\pi x\sin(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^\pi x\sin((n+1)x)dx - \frac{1}{\pi}\int_0^\pi x\sin((n-1)x)dx$$

Now, if $n > 1$, we have

$$\frac{1}{\pi}\int_0^\pi x\sin((n+1)x)dx = \frac{1}{\pi}\left(\Big[-x\frac{\cos((n+1)x)}{n+1}\Big]_0^\pi + \frac{1}{n+1}\underbrace{\int_0^\pi \cos((n+1)x)\,dx}_{=0}\right) = \frac{(-1)^n}{n+1}$$

Similarly,

$$\frac{1}{\pi}\int_0^\pi x\sin((n-1)x)dx = \frac{1}{\pi}\left(\Big[-x\frac{\cos((n-1)x)}{n-1}\Big]_0^\pi + \frac{1}{n-1}\underbrace{\int_0^\pi \cos((n-1)x)\,dx}_{=0}\right) = \frac{(-1)^n}{n-1}$$

If $n = 1$, we have

$$a_1 = \frac{1}{\pi}\int_0^\pi x\sin(2x)\,dx = \frac{(-1)^1}{1+1} = \frac{-1}{2}$$

Now, let us put everything together,

$$C(x) = 1 - \frac{\cos(x)}{2} + \sum_{n=2}^{\infty}\left(\frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1}\right)\cos(nx)$$

Note 1: The sum can be simplified further into

$$C(x) = 1 - \frac{\cos(x)}{2} + \sum_{n=2}^{\infty}\frac{2(-1)^{n+1}}{n^2-1}\cos(nx)$$

Note 2: It seems like we had to compute a lot of integrals but in fact they are all cousins (if not siblings or even twins), they are all in the form of

$$\int_0^\pi x\sin(ax)\,dx$$

(b) The same approach can be applied here

$$b_n = \frac{2}{\pi}\int_0^\pi x\sin(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^\pi x\cos((n-1)x)\,dx - \frac{1}{\pi}\int_0^\pi x\cos((n+1)x)\,dx$$

If $n > 1$, we have

$$\frac{1}{\pi}\int_0^\pi x\cos((n+1)x)\,dx = \frac{1}{\pi}\left(\underbrace{\left[\frac{x\sin((n+1)x)}{n+1}\right]_0^\pi}_{=0} - \frac{1}{n+1}\int_0^\pi \sin((n+1)x)\,dx\right) = -\frac{1+(-1)^n}{\pi(n+1)^2}$$

Similarly

$$\frac{1}{\pi}\int_0^\pi x\cos((n-1)x)\,dx = -\frac{1+(-1)^n}{\pi(n-1)^2}$$

Which means that

$$b_n = -\frac{1+(-1)^n}{\pi(n-1)^2} + \frac{1+(-1)^n}{\pi(n+1)^2}$$

Now, we only need to find b_1:

$$b_1 = \frac{2}{\pi}\int_0^\pi x\sin^2(x)\,dx = \frac{1}{\pi}\int_0^\pi x(1-\cos(2x))\,dx = \frac{2}{\pi}\int_0^\pi x dx - \frac{2}{\pi}\underbrace{\int_0^\pi x\cos(2x)\,dx}_{=0} = \frac{\pi}{2}$$

Finally we have

$$S(x) = \frac{\pi}{2}\sin(x) - \sum_{n=2}^{\infty}\left(\frac{1+(-1)^n}{\pi(n-1)^2} - \frac{1+(-1)^n}{\pi(n+1)^2}\right)\sin(nx)$$

Note: The previous comments apply here, the series can be simplified into

$$S(x) = \frac{\pi}{2}\sin(x) - \sum_{n=2}^{\infty}\frac{4n(1+(-1)^n)}{\pi(n^2-1)^2}\sin(nx)$$

Question: It looks like b_n converge to zero much faster than a_n, what is so special about f that made this happen? Is there a very quick way to see that $a_n = \mathcal{O}(n^{-2})$ and $b_n = \mathcal{O}(n^{-3})$ in this case?

Exercise 38. Let $f(x) = \cos(\pi x)$ on $(0,1)$.

(a) Find and simplify the cosine Fourier series of f on $(0,1)$.

(b) Find the sine Fourier series of f on $(0,1)$.

Solution 38. (a) It is easy to see that $a_0 = 0$ and that

$$a_n = 2\int_0^1 \cos(\pi x)\cos(n\pi x)\,dx = \int_0^1 \cos((n-1)\pi x)\,dx + \int_0^1 \cos((n+1)\pi x)\,dx$$

If $n > 1$ then $a_n = 0$, if $n = 1$, we have

$$a_1 = 2\int_0^1 \cos^2(\pi x)\,dx = \int_0^1 1 + \cos(2\pi x)\,dx = 1$$

This means that

$$C(x) = \cos(\pi x)$$

Note 1: There are at least three quick solutions to this problem:

- You can use the fact that the set $\{\cos(n\pi x)\}_{n=0}^{\infty}$ is orthogonal on $(0,1)$ which is a change of variables away from "the set $\{\cos(nx)\}_{n=0}^{\infty}$ is orthogonal on $(0,\pi)$".
- By comparing f to its Fourier series:

$$\frac{0}{2} + 1\cos(\pi x) + 0\cos(2x) + 0\cos(3x) + \cdots = \frac{a_0}{2} + a_1\cos(\pi x) + a_2\cos(2x) + a_3\cos(3x) + \ldots$$

- The cosine Fourier f on $(0,1)$ is a projection of f onto the subspace spanned by $\{\cos(n\pi x)\}_{n=0}^{\infty}$, since $f(x) = \cos(\pi x)$ is a basis of this subspace, then its projection is equal to itself.

(b)

$$b_n = 2\int_0^1 \cos(\pi x)\sin(n\pi x)\,dx = \int_0^1 \sin((n+1)\pi x)\,dx + \int_0^1 \sin((n-1)\pi x)\,dx$$

If $n > 1$, both integrals are easy to compute

$$b_n = \frac{1+(-1)^n}{\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) = \frac{2n(1+(-1)^n)}{\pi(n^2-1)}$$

and if $n = 1$, we have

$$b_1 = 2\int_0^1 \cos(\pi x)\sin(pix)\,dx = \int_0^1 \sin(2\pi x)\,dx = 0.$$

Therefore the sine series is

$$S(x) = \sum_{n=2}^{\infty} \frac{2n(1+(-1)^n)}{\pi(n^2-1)}\sin(n\pi x)$$

Question: It looks like this b_n converge slowly to 0, why is that?

Exercise 39. Let $f(x) = \cosh(x)$ on $(-\pi, \pi)$.

(a) Find the Fourier series of f on $(-\pi, \pi)$. Hint: use the fact that the derivative of cosh is sinh and the derivative of sinh is cosh.

(b) By choosing suitable values for x and Dirichlet's theorem, find the value of the following two series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$$

Solution 39. (a) The cosh function is even, so $b_n = 0$. We only need to find a_0 and a_n (we can use the parity of f to simplify our calculations).

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \cosh(x)\, dx = \frac{2}{\pi}\Big[\sinh(x)\Big] = \frac{2\sinh(\pi)}{\pi}.$$

To compute a_n, we need to evaluate the following integral

$$\begin{aligned}
I_n &= \int_0^{\pi} \cosh(x)\cos(nx)\, dx \\
&= \Big[\sinh(x)\cos(nx)\Big]_0^{\pi} + n\int_0^{\pi} \sinh(x)\sin(nx)\, dx \\
&= \sinh(\pi)(-1)^n + n\int_0^{\pi} \sinh(x)\sin(nx)\, dx \\
&= \sinh(\pi)(-1)^n + n\Big[\cosh(x)\sin(nx)\Big]_0^{\pi} - n^2\int_0^{\pi} \cosh(x)\cos(nx)\, dx \\
I_n &= \sinh(\pi)(-1)^n - n^2 I_n.
\end{aligned}$$

Which leads to $I_n = \frac{(-1)^n \sinh(pi)}{n^2+1}$ and $a_n = \frac{2}{\pi} I_n$. Finally, we have

$$\boxed{S(x) = \frac{\sinh(\pi)}{\pi} + \frac{2\sinh(\pi)}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}\cos(nx)}$$

(b) Notice that the function f and its periodic extension are continuous at 0 and π so we can apply Dirichlet's theorem easily to get

$$S(\pi) = f(\pi) = \cosh(\pi), \qquad S(0) = f(0) = 1.$$

So,

$$\cosh(\pi) = \frac{\sinh(\pi)}{\pi} + \frac{2\sinh(\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}(-1)^n = \frac{\sinh(\pi)}{\pi} + \frac{2\sinh(\pi)}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2+1}$$

Solving for the sum, we get

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2+1} = -\frac{1}{2} + \frac{\pi\cosh(\pi)}{2\sinh(\pi)}}.$$

The same technique can be applied to find the second sum

$$1 = \frac{\sinh(\pi)}{\pi} + \frac{2\sinh(\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1},$$

which would give you

$$\boxed{\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1} = -\frac{1}{2} + \frac{\pi}{2\sinh(\pi)}}$$

Note: If we extend these two sums to all integers (The set of all integers is denoted by the symbol $\mathbb{Z}$), we'll get the following beautiful formulas

$$\sum_{n\in\mathbb{Z}}\frac{1}{n^2+1} = \pi\coth(\pi), \qquad \sum_{n\in\mathbb{Z}}\frac{(-1)^n}{n^2+1} = \pi\,\mathrm{csch}(\pi).$$

Exercise 40. Let $f(x) = x(L-x)$ where L is a positive number.

1. Find the cosine series of f on $(0, L)$.
2. Find the sine series of f on $(0, L)$.

Solution 40. Note that the function was chosen in a such way to make integration by parts easy. Let us start with the cosine series

$$a_0 = \frac{2}{L}\int_0^L x(L-x)\,dx = \frac{2}{L}\left[\frac{Lx^2}{2} - \frac{x^3}{3}\right]_0^L = \frac{L^2}{3}$$

$$\begin{aligned}
a_n &= \frac{2}{L}\int_0^L x(L-x)\cos\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{2}{L}\underbrace{\left[x(L-x)\frac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\right]_0^L}_{=0} - \frac{2}{n\pi}\int_0^L (L-2x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\
&= -\frac{2}{n\pi}\int_0^L (L-2x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\
&= -\frac{2}{n\pi}\left[(L-2x)\frac{-\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\right]_0^L + \frac{4L}{n^2\pi^2}\underbrace{\int_0^L \cos\left(\frac{n\pi x}{L}\right)\,dx}_{=0} \\
&= \frac{-2L^2\left((-1)^n+1\right)}{n^2\pi^2}.
\end{aligned}$$

A similar approach can be applied to get the cosine series:

$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L x(L-x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{2}{L}\underbrace{\left[x(L-x)\frac{-\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\right]_0^L}_{=0} + \frac{2}{n\pi}\int_0^L (L-2x)\cos\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{2}{n\pi}\int_0^L (L-2x)\cos\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{2}{n\pi}\underbrace{\left[(L-2x)\frac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\right]_0^L}_{=0} + \frac{4L}{n^2\pi^2}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{4L}{n^2\pi^2}\left[\frac{-\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\right]_0^L \\
&= \frac{4L^2}{n^3\pi^3}\left(1-(-1)^n\right).
\end{aligned}$$

Now, we have

$$\boxed{C(x) = \frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{-2L^2\left((-1)^n + 1\right)}{n^2\pi^2} \cos\left(\frac{n\pi x}{L}\right)}$$

$$\boxed{Si(x) = \sum_{n=1}^{\infty} \frac{4L^2}{n^3\pi^3}\left(1 - (-1)^n\right) \sin\left(\frac{n\pi x}{L}\right)}$$

A shorter way: We could've applied the substitution $x = Lu$ to get

$$a_n = 2\int_0^1 Lu(L - Lu)\cos\left(n\pi u\right)\ du = 2L^2\int_0^1 u(1-u)\cos(n\pi u)\ du$$

Which is much nicer. The same can be applied to b_n.

Exercise 41. (*Reverse engineering a Fourier series*) Let f be a $2\pi-$periodic function, assume that for any x in $(-\pi, \pi)$:

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}.$$

Find a closed form expression (without sums) of $f(x)$. Hint: Find a closed form expression for $f(\pi - x)$, the notes can be helpful.

Sin

Solution 41. We have $\sin(n\pi - nx) = -(-1)^n \sin(nx)$ so

$$f(\pi - x) = \sum_{n=1}^{\infty} \frac{-(-1)^n}{n} \sin(nx)$$

which is the Fourier series of $\frac{x}{2}$ on $(-\pi, \pi)$ (see the notes). For $x \in (0, \pi)$ we have $\pi - x \in (0, \pi)$ and $f(\pi - x) = \frac{x}{2}$ which means that on $(0, \pi)$, we have $f(x) = \frac{\pi - x}{2}$, f is odd so on $(-\pi, 0)$, we have $f(x) = \frac{-\pi - x}{2}$.

Exercise 42. Let $f(x) = e^{-x}$ on $(0,1)$ and let $C(x), Si(x)$ be the Fourier cosine and sine series on $(0,1)$ respectively. What are the values of

$$C\left(-\frac{1}{2}\right), \qquad Si\left(-\frac{1}{2}\right).$$

Justify your answer.

Hint 1: You do not need to compute the coefficients.
Hint 2: Use Dirichlet's theorem and the properties of $C(x), Si(x)$.

Solution 42. f is continuous at $\frac{1}{2}$, so by Dirichlet's theorem:

$$C\left(\frac{1}{2}\right) = Si\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) = e^{-1/2} = \frac{1}{\sqrt{e}}.$$

Now, $C(x)$ is even (because it is a sum of cosines). Therefore

$$C\left(-\frac{1}{2}\right) = C\left(\frac{1}{2}\right) = \frac{1}{\sqrt{e}}.$$

On the other hand, $Si(x)$ is odd so

$$Si\left(-\frac{1}{2}\right) = -Si\left(\frac{1}{2}\right) = \frac{-1}{\sqrt{e}}.$$

Exercise 43. Let f be defined on $(-\pi, \pi)$ by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < \pi. \\ -1 & \text{if } -\pi < x < 0. \end{cases}$$

Find the Fourier series of f on $(-\pi, \pi)$.
Fun fact: The FS of f can be used to prove that

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots,$$

This formula is known as Leibniz' formula (Yes, the same Leibniz from calc 1).

Solution 43. f is odd, we only need to find b_n

$$b_n = \frac{2}{\pi} \int_0^{\pi} \sin(nx) = \frac{2}{\pi} \left[\frac{-\cos(nx)}{n} \right]_0^{\pi} = \frac{2(1 - \cos(n\pi))}{n\pi} = \frac{2(1 - (-1)^n)}{n\pi}$$

Therefore, the Fourier series of f is

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin(nx)$$

Exercise 44. Solve the following S-L problem

$$\begin{cases} y''(x) + \lambda y(x) = 0, \\ y(-L) = 0, \\ y(L) = 0. \end{cases}$$

Where L is some fixed positive number.

Solution 44.

- If $\lambda = 0$, then $y(x) = c_1 x + c_2$ but

$$\begin{cases} -c_1 L + c_2 = 0, \\ c_1 L + c_2 = 0, \end{cases}$$

leads to $c_1 = c_2 = 0$ because $L \neq 0$. Hence, $\lambda = 0$ is not an eigenvalue.

- If $\lambda < 0$, let $\lambda = -\alpha^2$, then $y(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$, plug it back into the boundary conditions (BC for short) to get

$$\begin{cases} c_1 \cosh(\alpha L) - c_2 \sinh(\alpha L) = 0 \\ c_1 \cosh(\alpha L) + c_2 \sinh(\alpha L) = 0 \end{cases}$$

Since $\sinh(\alpha L) \neq 0$ and $\cosh(\alpha L) \neq 0$, we would get $c_1 = c_2 = 0$ (by taking the sum/difference of the two equations). Hence, there are no negative eigenvalues.

- If $\lambda > 0$, let $\lambda = \alpha^2$, then $y(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$, plug it back into the boundary conditions (BC for short) to get

$$\begin{cases} c_1 \cos(\alpha L) - c_2 \sin(\alpha L) = 0 \\ c_1 \cos(\alpha L) + c_2 \sin(\alpha L) = 0 \end{cases}$$

There are at least two ways to proceed here

  - Compute the determinant of the system and set it equal to 0, you will get $\sin(2\alpha L) = 0$, which means that $\alpha = \frac{n\pi}{2L}$, now if $n = 2k$, the system can be reduced to just $c_1 = 0$. If $n = 2k - 1$, the system can be reduced to $c_2 = 0$. Hence, our eigenvalues/eigenfunctions are

$$(\lambda_{2k}, y_{2k}(x)) = \left(\left(\frac{k\pi}{L} \right)^2, \sin\left(\frac{k\pi x}{L} \right) \right)$$

and

$$(\lambda_{2k-1}, y_{2k-1}(x)) = \left(\left(\frac{(2k-1)\pi}{2L} \right)^2, \cos\left(\frac{(2k-1)\pi x}{2L} \right) \right).$$

  - You can intuitively see that if we assume that c_1 and c_2 are not simultaneously zero, then ($c_1 = 0$ and $\sin(\alpha L) = 0$) OR ($c_2 = 0$ and $\cos(\alpha L) = 0$). You will reach the same conclusion.

Exercise 45. Solve the following S-L problem

$$\begin{cases} y''(x) + \lambda y(x) = 0, \\ y'(0) = 0, \\ y(L) = 0. \end{cases}$$

Solution 45.

- If $\lambda = 0$, then $y(x) = c_1 x + c_2$ but

$$\begin{cases} c_1 = 0, \\ c_1 L + c_2 = 0, \end{cases}$$

leads to $c_1 = c_2 = 0$. Hence, $\lambda = 0$ is not an eigenvalue.

- If $\lambda < 0$, let $\lambda = -\alpha^2$, then $y(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$, plug it back into the boundary conditions (BC for short) to get

$$\begin{cases} \alpha c_2 = 0 \\ c_1 \cosh(\alpha L) + c_2 \sinh(\alpha L) = 0 \end{cases}$$

Which leads to $c_1 = c_2 = 0$. Hence, there are no negative eigenvalues.

- If $\lambda > 0$, let $\lambda = \alpha^2$, then $y(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$, plug it back into the boundary conditions to get

$$\begin{cases} \alpha c_2 = 0 \\ c_1 \cos(\alpha L) + c_2 \sin(\alpha L) = 0 \end{cases}$$

Which leads to $c_2 = 0$ and $\cos(\alpha L) = 0$, this means that $\alpha = \frac{(2n-1)\pi}{2L}$. Hence the set of eigenvalues/eigenfunctions is

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2, \qquad .y_n(x) = \cos\left(\frac{(2n-1)\pi}{2L}x\right).$$

Exercise 46. Show that the following SL problem has a negative eigenvalue

$$\begin{cases} y''(x) + \lambda y(x) = 0, \\ y'(0) = 0, \\ y(1) - y'(1) = 0. \end{cases}$$

Hint: Only study the case $\lambda < 0$, use the intermediate value theorem.

Note 1: The IVT state that if f is continuous and there are a, b such that $f(a)$ and $f(b)$ have different signs, then there is c between a and b such that $f(c) = 0$.

Note 2: In order to find some suitable values for a, b, you may want to use Mathematica to plot f.

Solution 46. Let $\lambda = -\alpha^2$ where $\alpha > 0$, then

$$y(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x).$$

The condition $y'(0) = 0$ gives $c_2 = 0$. The second condition is equivalent to

$$c_1 \underbrace{(\cosh(\alpha) - \alpha \sinh(\alpha))}_{=g(\alpha)} = 0$$

Notice that g is continuous, $g(0) = 1 > 0$, and $g(2) = \frac{3-e^4}{2e^2} < 0$. Therefore, there is some $\alpha_0 \in (0, 2)$ such that $g(\alpha_0) = 0$.

Now, $\lambda_0 = -\alpha_0^2$ is a negative eigenvalue of the given SL problem.

Exercise 47. (From ordinary to fantastic) Consider the following SL problem

$$\begin{cases} y''(x) + \lambda y(x) = 0, & (1) \\ y(0) = y(2) = 0. & (2) \end{cases}$$

Note that this is not one of the fantastic four. Instead of brute-forcing this problem, we will use a different tactic: Let $z(x) = y(2x)$, then

$$z(0) = y(0) = 0 \qquad \text{and} \qquad z(1) = y(2) = 0. \tag{3}$$

(a) Notice that since equation (1) holds for all x, we can replace x by $2x$ and get $y''(2x) + \lambda y(2x) = 0$. Using this observation, find μ in terms of λ such that

$$z''(x) + \mu z(x) = 0.$$

(b) At this step, we can see that z satisfies an SL problem:

$$\begin{cases} z''(x) + \mu z(x) = 0, & \text{(from part (a))} \\ z(0) = z(1) = 0. & \text{(from (3))} \end{cases}$$

Using the notes (or exercise 1), list all eigen-pairs (μ_n, z_n).

(c) Using the relationships between (μ, z) and (λ, y). Find the eigen-pairs (λ_n, y_n).

Note: This process avoids having to go through the three cases for λ since z is the solution to a fantastic SL problem. Feel free to try it on example 3 on page 11 in the notes to see how you can obtain a neater solution effortlessly.

Solution 47. (a) Take $y(2x) = z(x)$, differentiate both sides twice (using the chain rule) to get

$$4y''(2x) = z''(x)$$

Now, let us go back to $y''(2x) + \lambda y(2x) = 0$, we can replace $y''(2x)$ by $\frac{1}{4}z''(x)$ and we can replace $y(2x)$ by $z(x)$ to obtain

$$\frac{1}{4}z''(x) + \lambda z(x) = 0$$

Now, we multiply both sides by 4 to get

$$z''(x) + 4\lambda z(x) = 0.$$

From here, we can see that $\mu = 4\lambda$.

(b) From (SL1), we have

$$z_n(x) = \sin(n\pi x), \qquad \mu_n = n^2\pi^2.$$

(c) We have $\lambda_n = \dfrac{1}{4}\mu_n = \dfrac{n^2\pi^2}{4}$ and

$$y_n(x) = z_n\left(\frac{x}{2}\right) = \sin\left(\frac{n\pi x}{2}\right).$$

Exercise 48. Solve the following S-L problem

$$\begin{cases} y''(x) + \lambda y(x) = 0, \\ y'(-2) = 0, \\ y(2) = 0. \end{cases}$$

Solution 48.

- If $\lambda = 0$, then $y(x) = c_1 x + c_2$ but the BC are

$$\begin{cases} c_1 = 0, \\ 3c_1 + c_2 = 0, \end{cases}$$

leads to $c_1 = c_2 = 0$ because $L \neq 0$. Hence, $\lambda = 0$ is not an eigenvalue.

- If $\lambda < 0$, let $\lambda = -\alpha^2$, then $y(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$, plug it back into the boundary conditions to get

$$\begin{cases} -c_1 \sinh(2\alpha) + c_2 \cosh(2\alpha) = 0 \\ c_1 \cosh(2\alpha) + c_2 \sinh(2\alpha) = 0 \end{cases}$$

The determinant of this system is $-\cosh^2(2\alpha) - \sinh^2(2\alpha) = -\cosh(4\alpha) \neq 0$. Therefore, the only solution is $c_1 = c_2 = 0$ which means that there are no negative eigenvalues.

- If $\lambda > 0$, let $\lambda = \alpha^2$, then $y(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$, plug it back into the boundary conditions (BC for short) to get

$$\begin{cases} c_1 \sin(2\alpha) + c_2 \cos(2\alpha) = 0 \\ c_1 \cos(2\alpha) + c_2 \sin(2\alpha) = 0 \end{cases}$$

The determinant of this system is $\sin^2(2\alpha) - \cos^2(2\alpha) = -\cos(4\alpha)$. If you set the determinant equal to 0, you'd get $\alpha = \frac{(2n-1)\pi}{8}$. Plug this into the system to get

$$\begin{cases} c_1 \sin\left(\frac{(2n-1)\pi}{4}\right) + c_2 \cos\left(\frac{(2n-1)\pi}{4}\right) = 0 \\ c_1 \cos\left(\frac{(2n-1)\pi}{4}\right) + c_2 \sin\left(\frac{(2n-1)\pi}{4}\right) = 0 \end{cases}$$

If you take a quick look at $\cos\left(\frac{(2n-1)\pi}{4}\right)$ and $\sin\left(\frac{(2n-1)\pi}{4}\right)$, you will see that when n is even, we have $\cos\left(\frac{(2n-1)\pi}{4}\right) = -\sin\left(\frac{(2n-1)\pi}{4}\right)$ which means that $c_1 = c_2$ and when n is odd, we have $\cos\left(\frac{(2n-1)\pi}{4}\right) = \sin\left(\frac{(2n-1)\pi}{4}\right)$ which means that $c_1 = -c_2$. Let us put all of that together to find

$$\lambda_{2k} = \left(\frac{(4k-1)\pi}{8}\right)^2, \qquad .y_{2k}(x) = \cos\left(\frac{(4k-1)\pi}{8}x\right) + \sin\left(\frac{(4k-1)\pi}{8}x\right),$$

and

$$\lambda_{2k-1} = \left(\frac{(4k-3)\pi}{8}\right)^2, \qquad .y_{2k}(x) = \cos\left(\frac{(4k-3)\pi}{8}x\right) - \sin\left(\frac{(4k-3)\pi}{8}x\right).$$

Exercise 49. Solve the following differential equation

$$\begin{cases} y''(x) + \lambda y(x) = 0, \\ y(-\pi) = y(\pi), \\ y'(-\pi) = y'(\pi). \end{cases}$$

Note 1: This is not a *regular* Sturm-Liouville problem. However, it can be solved similarly.

Note 2: This kind of boundary conditions is called periodic boundary conditions. This kind of problems arises when you are studying a differential equation on a circle (you can think of a circle as a segment with the two endpoints identified as the same point).

Solution 49.

- If $\lambda = 0$, then $y(x) = c_1 x + c_2$, the BC can be written as

$$\begin{cases} -c_1\pi + c_2 = c_1\pi + c_2, \\ c_1 = c_1, \end{cases}$$

leads to $c_1 = 0$. Hence, $\lambda_0 = 0$ **is an eigenvalue** associated with $y_0(x) = 1$.

- If $\lambda < 0$, let $\lambda = -\alpha^2$, then $y(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$, plug it back into the boundary conditions (BC for short) to get

$$\begin{cases} c_1 \cosh(\alpha\pi) - c_2 \sinh(\alpha\pi) = c_1 \cosh(\alpha\pi) + c_2 \sinh(\alpha\pi) \\ \alpha\left(-c_1 \sinh(\alpha\pi) + c_2 \cosh(\alpha\pi)\right) = \alpha\left(c_1 \sinh(\alpha\pi) + c_2 \cosh(\alpha\pi)\right) \end{cases}$$

Which leads to $c_1 = c_2 = 0$. Hence, there are no negative eigenvalues.

- If $\lambda > 0$, let $\lambda = \alpha^2$, then $y(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$, plug it back into the boundary conditions to get

$$\begin{cases} c_1 \cos(\alpha\pi) - c_2 \sin(\alpha\pi) = c_1 \cos(\alpha\pi) + c_2 \sin(\alpha\pi) \\ \alpha\left(-c_1 \sin(\alpha\pi) + c_2 \cos(\alpha\pi)\right) = \alpha\left(c_1 \sin(\alpha\pi) + c_2 \cos(\alpha\pi)\right) \end{cases}$$

Which can be simplified to

$$\begin{cases} c_2 \sin(\alpha\pi) = 0, \\ c_1 \sin(\alpha\pi) = 0. \end{cases}$$

Since $(c_1, c_2) \neq (0, 0)$, we have $\alpha = n$. Which means that $\lambda_n = n^2$ and $y_n(x) = c_1 \cos(nx) + c_2 \sin(nx)$. This can broken down into

$$(\lambda_{2k}, y_{2k}(x)) = \left(k^2, \sin(kx)\right), \quad (\lambda_{2k-1}, y_{2k-1}(x)) = \left(k^2, \cos(kx)\right).$$

Note 1: Notice that the eigenspace associated with each positive eigenvalue k has dimension 2. This would not have happened if this was a Sturm-Liouville problem (see the second property on page 5 of lesson 15).

Note 2: These eigenfunctions (the constant eigenfunction is included) are the basis functions of Fourier series.

Exercise 50. Solve the following S-L problem

$$\begin{cases} y''(x) + \lambda y(x) = 0, \\ y'(-3) = 0, \\ y(3) = 0. \end{cases}$$

Solution 50.

- If $\lambda = 0$, then $y(x) = c_1 x + c_2$ but the BC are

$$\begin{cases} c_1 = 0, \\ 3c_1 + c_2 = 0, \end{cases}$$

leads to $c_1 = c_2 = 0$ because $L \neq 0$. Hence, $\lambda = 0$ is not an eigenvalue.

- If $\lambda < 0$, let $\lambda = -\alpha^2$, then $y(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$, plug it back into the boundary conditions to get

$$\begin{cases} -c_1 \sinh(3\alpha) + c_2 \cosh(3\alpha) = 0 \\ c_1 \cosh(3\alpha) + c_2 \sinh(3\alpha) = 0 \end{cases}$$

The determinant of this system is $-\cosh^2(3\alpha) - \sinh^2(3\alpha) = -\cosh(6\alpha) \neq$. Therefore, the only solution is $c_1 = c_2 = 0$ which means that there are no negative eigenvalues.

- If $\lambda > 0$, let $\lambda = \alpha^2$, then $y(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$, plug it back into the boundary conditions (BC for short) to get

$$\begin{cases} c_1 \sin(3\alpha) + c_2 \cos(3\alpha) = 0 \\ c_1 \cos(3\alpha) + c_2 \sin(3\alpha) = 0 \end{cases}$$

The determinant of this system is $\sin^2(3\alpha) - \cos^2(3\alpha) = -\cos(6\alpha)$. If you set the determinant equal to 0, you'd get $\alpha = \frac{(2n-1)\pi}{12}$. Plug this into the system to get

$$\begin{cases} c_1 \sin\left(\frac{(2n-1)\pi}{4}\right) + c_2 \cos\left(\frac{(2n-1)\pi}{4}\right) = 0 \\ c_1 \cos\left(\frac{(2n-1)\pi}{4}\right) + c_2 \sin\left(\frac{(2n-1)\pi}{4}\right) = 0 \end{cases}$$

If you take a quick look at $\cos\left(\frac{(2n-1)\pi}{4}\right)$ and $\sin\left(\frac{(2n-1)\pi}{4}\right)$, you will see that when n is even, we have $\cos\left(\frac{(2n-1)\pi}{4}\right) = -\sin\left(\frac{(2n-1)\pi}{4}\right)$ which means that $c_1 = c_2$ and when n is odd, we have $\cos\left(\frac{(2n-1)\pi}{4}\right) = \sin\left(\frac{(2n-1)\pi}{4}\right)$ which means that $c_1 = -c_2$. Let us put all of that together to find

$$\lambda_{2k} = \left(\frac{(4k-1)\pi}{12}\right)^2, \qquad .y_{2k}(x) = \cos\left(\frac{(4k-1)\pi}{12}x\right) + \sin\left(\frac{(4k-1)\pi}{12}x\right),$$

and

$$\lambda_{2k-1} = \left(\frac{(4k-3)\pi}{12}\right)^2, \qquad .y_{2k}(x) = \cos\left(\frac{(4k-3)\pi}{12}x\right) - \sin\left(\frac{(4k-3)\pi}{12}x\right).$$

Exercise 51. Use separation of variables to find a family of solutions to the following PDE

$$u_t(x,t) - c^2 u_{xx}(x,t) + Ru(x,t) = 0.$$

Hint: See lesson 16, the last example is a bit similar.
Note: This is known as the reaction-diffusion equation and has many applications in physics, engineering and economics.

Solution 51. Assume that $u(x,t) = X(x)T(t)$, then

$$X(x)T'(t) - c^2 X''(x)T(t) + RX(x)T(t) = 0$$

divide both sides by $X(t)T(t)$ to get

$$\frac{T'(t)}{T(t)} - c^2 \frac{X''(x)}{X(x)} + R = 0$$

We can re-arrange it as

$$\frac{T'(t)}{T(t)} + R - c^2 \frac{X''(x)}{X(x)} = 0.$$

From here, there are multiple ways to proceed

1. Method 1: Separate the equation in the following way

$$\frac{T'(t)}{T(t)} + R = c^2 \frac{X''(x)}{X(x)} = -\lambda.$$

Now, we have $T'(t) = -(\lambda + R)T(t)$ which means that $T(t) = e^{-(\lambda+R)t}$. We also have $X''(x) + \frac{\lambda}{c^2}X(x) = 0$ which gives us three cases

$$\begin{cases} \lambda = 0, & X(x) = ax + b, & u(x,t) = (ax+b)e^{-Rt}. \\ \lambda > 0, & X(x) = a\cos\left(\frac{x\sqrt{\lambda}}{c}\right) + b\sin\left(\frac{x\sqrt{\lambda}}{c}\right), & u(x,t) = \left(a\cos\left(\frac{x\sqrt{\lambda}}{c}\right) + b\sin\left(\frac{x\sqrt{\lambda}}{c}\right)\right)e^{-(R+\lambda)t} \\ \lambda < 0, & X(x) = a\cosh\left(\frac{x\sqrt{-\lambda}}{c}\right) + b\sinh\left(\frac{x\sqrt{-\lambda}}{c}\right), & u(x,t) = \left(a\cosh\left(\frac{x\sqrt{-\lambda}}{c}\right) + b\sinh\left(\frac{x\sqrt{-\lambda}}{c}\right)\right)e^{-(R+\lambda)t} \end{cases}$$

2. Method 2: Separate the equation in the following way

$$\frac{T'(t)}{c^2 T(t)} + \frac{R}{c^2} = \frac{X''(x)}{X(x)} = -\lambda.$$

Now, we have $T'(t) = -c^2(\lambda + R)T(t)$ which means that $T(t) = e^{-c^2(\lambda+R)t}$. We also have $X''(x) + \lambda X(x) = 0$ which gives us three cases

$$\begin{cases} \lambda = 0, & X(x) = ax + b, & u(x,t) = (ax+b)e^{-c^2Rt}. \\ \lambda > 0, & X(x) = a\cos\left(x\sqrt{\lambda}\right) + b\sin\left(x\sqrt{\lambda}\right), & u(x,t) = \left(a\cos\left(x\sqrt{\lambda}\right) + b\sin\left(x\sqrt{\lambda}\right)\right)e^{-c^2(R+\lambda)t} \\ \lambda < 0, & X(x) = a\cosh\left(x\sqrt{-\lambda}\right) + b\sinh\left(x\sqrt{-\lambda}\right), & u(x,t) = \left(a\cosh\left(x\sqrt{-\lambda}\right) + b\sinh\left(x\sqrt{-\lambda}\right)\right)e^{-c^2(R+\lambda)t} \end{cases}$$

3. Method 3: Separate the equation in the following way

$$\frac{T'(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)} - R = -\lambda.$$

Now, we have $T'(t) = -\lambda T(t)$ which means that $T(t) = e^{-\lambda t}$. We also have $X''(x) + \frac{\lambda - R}{c^2} X(x) = 0$ which gives us three cases

$$\begin{cases} \lambda = R, & X(x) = ax + b, & u(x,t) = (ax+b)e^{-Rt}. \\ \lambda > R, & X(x) = a\cos\left(\frac{x\sqrt{\lambda - R}}{c}\right) + b\sin\left(\frac{x\sqrt{\lambda - R}}{c}\right), & u(x,t) = \left(a\cos\left(\frac{x\sqrt{\lambda - R}}{c}\right) + b\sin\left(\frac{x\sqrt{\lambda - R}}{c}\right)\right)e^{-\lambda t} \\ \lambda < R, & X(x) = a\cosh\left(\frac{x\sqrt{R - \lambda}}{c}\right) + b\sinh\left(\frac{x\sqrt{R - \lambda}}{c}\right), & u(x,t) = \left(a\cosh\left(\frac{x\sqrt{R - \lambda}}{c}\right) + b\sinh\left(\frac{x\sqrt{R - \lambda}}{c}\right)\right)e^{-\lambda t} \end{cases}$$

4. Method 4: Separate the equation in the following way

$$\frac{T'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} - \frac{R}{c^2} = -\lambda.$$

details omitted.

Exercise 52. Solve the following heat equation:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

Solution 52. The associated SL problem is

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X(0) = X(1). \end{cases}$$

The eigen-pairs are $(n^2\pi^2, \sin(n\pi x))$, so the solution can be written as

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) e^{-n^2\pi^2 t},$$

where

$$c_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

Exercise 53. Solve the following heat equation:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) \\ u_x(0,t) = u_x(1,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

Solution 53. The associated SL problem is

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X'(0) = X'(1). \end{cases}$$

The eigen-pairs are $(n^2\pi^2, \cos(n\pi x))$ and $(0, 1)$, so the solution can be written as

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\pi x) e^{-n^2\pi^2 t},$$

where

$$c_n = 2\int_0^1 f(x)\cos(n\pi x)\, dx$$

and

$$c_0 = \int_0^1 f(x)dx$$

Exercise 54. Solve the following wave equation:

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = -f(x) \\ u_t(x,0) = f(x) \end{cases}$$

Solution 54. The associated SL problem is

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X(0) = X(1). \end{cases}$$

The eigen-pairs are $(n^2\pi^2, \sin(n\pi x))$, so the solution can be written as

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos(n\pi t) + B_n \sin(n\pi t)\right) \sin(n\pi x),$$

where

$$A_n = 2\int_0^1 -f(x)\sin(n\pi x)\, dx$$

and

$$B_n = \frac{2}{n\pi}\int_0^1 f(x)\sin(nx)dx$$

Exercise 55. Solve the following Laplace equation in the square $[0,1] \times [0,1]$:

$$\begin{cases} u_{xx}(x,y) + u_{yy}(x,y) = 0, \\ u(0,y) = 0, \\ u(1,y) = 0, \\ u(x,1) = f(x), \\ u_y(x,0) = u(x,0). \end{cases}$$

Solution 55. We start with separation of variables $u(x,y) = X(x)Y(y)$. In this problem, X satisfies the following SL problem

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X(0) = X(1). \end{cases}$$

The eigen-pairs are $(n^2\pi^2, \sin(n\pi x))$. Now, we can write the solution as

$$u(x,y) = \sum_{n=1}^{\infty} \left(A_n \cosh(n\pi y) + B_n \sinh(n\pi y)\right) \sin(n\pi x).$$

The last BC gives us:

$$\sum_{n=1}^{\infty} n\pi B_n \sin(n\pi x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

which yields $A_n = n\pi B_n$. At this point, the solution looks like

$$u(x,y) = \sum_{n=1}^{\infty} B_n \left(n\pi \cosh(n\pi y) + \sinh(n\pi y)\right) \sin(n\pi x)$$

Plug in 1 for y and use the third boundary condition to get

$$f(x) = \sum_{n=1}^{\infty} B_n \left(n\pi \cosh(n\pi) + \sinh(n\pi)\right) \sin(n\pi x)$$

which is a sine Fourier series. Hence,

$$B_n \left(n\pi \cosh(n\pi) + \sinh(n\pi)\right) = 2\int_0^1 f(x) \sin(n\pi x)\, dx$$

Equivalently,

$$B_n = \frac{2}{\pi n \left(n\pi \cosh(n\pi) + \sinh(n\pi)\right)} \int_0^1 f(x) \sin(n\pi x)\, dx$$

Exercise 56. Solve the following Laplace equation in the rectangle $[0,1] \times [0,1]$:

$$\begin{cases} u_{xx}(x,y) + u_{yy}(x,y) = 0, \\ u(0,y) = 0, \\ u(1,y) = 0, \\ u(x,0) = f(x), \\ u_y(x,1) = 0. \end{cases}$$

Solution 56. The eigenfunctions are $\sin(n\pi x)$. The solution can be written as

$$u(x,y) = \sum_{n=1}^{\infty} \sin(n\pi x) \left[A_n \cosh(n\pi y) + B_n \sinh(n\pi y)\right]$$

We still have two boundary conditions to enforce. Let us start with the last one $u_y(x,1) = 0$, which translates to

$$0 = \sum_{n=1}^{\infty} \sin(n\pi x) \left[n\pi A_n \sinh(n\pi) + n\pi B_n \cosh(n\pi)\right].$$

This means that $B_n = -A_n \tanh(n\pi)$. Finally, to find A_n, we plug in $y = 0$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\, dx \Longrightarrow A_n = 2\int_0^1 f(x) \sin(n\pi x)\, dx$$

If you want to put everything together nicely, you could do :

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \left[\cosh(n\pi y) - \tanh(n\pi) \sinh(n\pi y)\right].$$

You could apply a hyperbolic trig identity to get

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \frac{\cosh((1-y)n\pi)}{\cosh(n\pi)}.$$

Exercise 57. Solve the following Laplace equation in the rectangle $[0,4]\times[0,1]$:

$$\begin{cases} u_{xx}(x,y)+u_{yy}(x,y)=0, \\ u(0,y)=0, \\ u(4,y)=0, \\ u(x,1)=f(x), \\ u_y(x,0)=u(x,0). \end{cases}$$

Solution 57. We proceed as usual, set $u(x,y)=X(x)Y(y)$, you will get the same SL problem with $X_n(x)=\sin\left(\frac{n\pi x}{4}\right)$, the only difference is that now we have

$$Y_n(y)=A_n\cosh\left(\frac{n\pi y}{4}\right)+B_n\sinh\left(\frac{n\pi y}{4}\right)$$

because $Y_n''(y)-\lambda_n y_n=0$. Now, we write u as

$$u(x,y)=\sum_{n=1}^{\infty}\left[A_n\cosh\left(\frac{n\pi y}{4}\right)+B_n\sinh\left(\frac{n\pi y}{4}\right)\right]\sin\left(\frac{n\pi x}{4}\right)$$

Now, we use the condition $u_y(x,0)=u(x,0)$ to get

$$\sum_{n=1}^{\infty}\frac{n\pi}{4}B_n\sin\left(\frac{n\pi x}{4}\right)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{4}\right)$$

Which means that $A_n=\frac{n\pi}{4}B_n$. Our solution looks like:

$$u(x,y)=\sum_{n=1}^{\infty}\left[\frac{n\pi}{4}\cosh\left(\frac{n\pi y}{4}\right)+\sinh\left(\frac{n\pi y}{4}\right)\right]B_n\sin\left(\frac{n\pi x}{4}\right)$$

The only remaining condition is $u(x,1)=f(x)$ which gives

$$f(x)=\sum_{n=1}^{\infty}\left[\frac{n\pi}{4}\cosh\left(\frac{n\pi}{4}\right)+\sinh\left(\frac{n\pi}{4}\right)\right]B_n\sin\left(\frac{n\pi x}{4}\right)$$

By comparing coefficients, we get

$$B_n=\frac{1}{\frac{n\pi}{4}\cosh\left(\frac{n\pi}{4}\right)+\sinh\left(\frac{n\pi}{4}\right)}\cdot 2\underbrace{\int_0^1 f(x)\sin\left(\frac{n\pi x}{4}\right)\,dx}_{=c_n}.$$

Finally, we have

$$u(x,y)=\sum_{n=1}^{\infty}\left[\frac{\frac{n\pi}{4}\cosh\left(\frac{n\pi y}{4}\right)+\sinh\left(\frac{n\pi y}{4}\right)}{\frac{n\pi}{4}\cosh\left(\frac{n\pi}{4}\right)+\sinh\left(\frac{n\pi}{4}\right)}\right]c_n\sin\left(\frac{n\pi x}{4}\right).$$

Note: We could impose the following condition $Y_n'(0)=Y_n(0)$ which should give us $A_n=\frac{n\pi}{4}B_n$ (the same thing that we got).

Exercise 58. Solve the following Laplace equation in the rectangle $[0,1] \times [0,1]$:

$$\begin{cases} u_{xx}(x,y) + u_{yy}(x,y) = 0, \\ u(0,y) = f(y), \\ u(1,y) = 2f(y), \\ u(x,0) = 0, \\ u_y(x,1) = 0. \end{cases}$$

Solution 58. This is a little different in the sense that the eigenfunctions are functions of y:

$$Y_n(y) = \sin\left(\frac{2n-1}{2}\pi y\right)$$

Therefore, the solution is

$$u(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{2n-1}{2}\pi y\right)\left[A_n \cosh\left(\frac{2n-1}{2}\pi x\right) + B_n \sinh\left(\frac{2n-1}{2}\pi x\right)\right].$$

The condition $u(0,y) = f(y)$ gives us

$$A_n = 2\int_0^1 f(y)\sin\left(\frac{2n-1}{2}\pi y\right)\, dy$$

The other boundary condition $u(1,y) = 2f(y)$ is more interesting since it gives us

$$2f(y) = \sum_{n=1}^{\infty} \sin\left(\frac{2n-1}{2}\pi y\right)\left[A_n \cosh\left(\frac{2n-1}{2}\pi\right) + B_n \sinh\left(\frac{2n-1}{2}\pi\right)\right]$$

We have

$$A_n \cosh\left(\frac{2n-1}{2}\pi\right) + B_n \sinh\left(\frac{2n-1}{2}\pi\right) = 4\int_0^1 f(y)\sin\left(\frac{2n-1}{2}\pi y\right)\, dy$$

We solve for B_n

$$B_n = \frac{1}{\sinh\left(\frac{2n-1}{2}\pi\right)}\left(4\int_0^1 f(y)\sin\left(\frac{2n-1}{2}\pi y\right)\, dy - A_n \cosh\left(\frac{2n-1}{2}\pi\right)\right)$$

We can do some sneaky simplifications since the integral is similar to A_n:

$$B_n = \frac{A_n}{\sinh\left(\frac{2n-1}{2}\pi\right)}\left(2 - \cosh\left(\frac{2n-1}{2}\pi\right)\right)$$

Exercise 59. Solve the following heat equation:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) \\ u(0,t) = 1 - e^{-t} \\ u(1,t) = e^t, \\ u(x,0) = 0 \end{cases}$$

Solution 59. We first start by eliminating the NHBC: we set $u = v + \psi$, where

$$\psi(x,t) = (1-x)(1-e^{-t}) + xe^t.$$

When we plug $v = u - \psi$ back we get

$$\begin{cases} v_t(x,t) = v_{xx}(x,t) - xe^t - (1-x)e^{-t} \\ v(0,t) = 0 \\ v(1,t) = 0, \\ v(x,0) = -x. \end{cases}$$

We can proceed in two ways:

- Method 1: we write the IC and the NHT as a sine series:

  $$-xe^t - (1-x)e^{-t} = \sum_{n=1}^{\infty} \beta_n(t) \sin(n\pi x), \qquad -x = \sum_{n=1}^{\infty} c_n \sin(n\pi x).$$

  The coefficients can be computed using :

  $$\beta_n(t) = \frac{2(-1)^n e^t - 2e^{-t}}{\pi n}, \qquad c_n = \frac{2(-1)^n}{\pi n}.$$

  We can express v as a series as well:

  $$v(x,t) = \sum_{n=1}^{\infty} \alpha_n(t) \sin(n\pi x).$$

  where the α_n's are solutions to the following IVP

  $$\begin{cases} \alpha_n'(t) + n^2\pi^2\alpha_n(t) = \beta_n(t), \\ \alpha_n(0) = c_n. \end{cases}$$

  The solution is given by

  $$\alpha_n(t) = c_n e^{-n^2\pi^2 t} + \int_0^t e^{-n^2\pi^2(t-r)}\beta_n(r)\, dr$$

  $$= \frac{2(-1)^n}{\pi n} e^{-n^2\pi^2 t} + \frac{2\left(e^{-t} - e^{\pi^2(-n^2)t}\right)}{\pi n - \pi^3 n^3} + \frac{2(-1)^n\left(e^t - e^{\pi^2(-n^2)t}\right)}{\pi^3 n^3 + \pi n}$$

- Method 2: We write v as $z + w$, where

$$\begin{cases} z_t(x,t) = z_{xx}(x,t) \\ z(0,t) = 0 \\ z(1,t) = 0, \\ z(x,0) = -x. \end{cases} \qquad \begin{cases} w_t(x,t) = w_{xx}(x,t) - xe^t + (1-x)e^{-t} \\ w(0,t) = 0 \\ w(1,t) = 0, \\ w(x,0) = 0. \end{cases}$$

We solve z using lesson 17 techniques and we solve w as a heat equation with a non-homogeneous term.

Exercise 60. Solve the following heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + x\sin(t), \\ u(0,t) = t, \\ u(1,t) = 0, \\ u(x,0) = x^2. \end{cases}$$

Solution 60. We eliminate the NHBC by using $u = v + \psi$, where $\psi = (1-x)t$. We get

$$\begin{cases} v_t(x,t) = v_{xx}(x,t) + x\sin(t) - (1-x), \\ v(0,t) = 0, \\ v(1,t) = 0, \\ v(x,0) = x^2. \end{cases}$$

And we proceed as before:

$$x\sin(t) - (1-x) = \sum_{n=1}^{\infty} \beta_n(t)\sin(n\pi x), \qquad x^2 = \sum_{n=1}^{\infty} c_n \sin(n\pi x).$$

We have

$$\beta_n(t) = -\frac{(-1)^n \sin(t) + 1}{\pi n}, \qquad c_n = -\frac{\pi^2(-1)^n n^2 - 2(-1)^n + 2}{\pi^3 n^3}, \qquad (\text{✸})$$

Now, we express v as

$$v(x,t) = \sum_{n=1}^{\infty} \alpha_n(t)\sin(n\pi x).$$

where

$$\begin{cases} \alpha_n'(t) + n^2\pi^2\alpha_n(t) = \beta_n(t), \\ \alpha_n(0) = c_n. \end{cases}$$

The solution is given by:

$$\alpha_n(t) = c_n e^{-n^2\pi^2 t} + \int_0^t e^{-n^2\pi^2(t-r)}\beta_n(r)\,dr$$

$$= -\frac{\pi^2(-1)^n n^2 - 2(-1)^n + 2}{\pi^3 n^3} e^{-n^2\pi^2 t} + \frac{e^{-n^2\pi^2 t} - 1}{\pi^3 n^3} + \frac{(-1)^{n+1}\left(e^{-n^2\pi^2 t} + \pi^2 n^2 \sin(t) - \cos(t)\right)}{\pi^5 n^5 + \pi n}$$

.

Exercise 61. Solve the following heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) - 1 + x - x\cos(t), \\ u(0,t) = 0, \\ u(1,t) = 0, \\ u(x,0) = x(1-x). \end{cases}$$

Note: This is an odd numbered problem from the textbook, you can find the final solution there and compare it to yours. There is a small typo there, the exact solution is:

$$\begin{aligned} u(x,t) = &\sum_{n=1}^{\infty} \frac{2}{n\pi}\left[\frac{-1}{n^2\pi^2} + (-1)^n \frac{n^2\pi^2\cos(t) + \sin(t)}{n^4\pi^4 + 1}\right]\sin(n\pi x) \\ &+ \sum_{n=1}^{\infty}\left[\frac{6 - 4(-1)^n}{n^3\pi^3} - (-1)^n \frac{2n\pi}{n^4\pi^4+1}\right] e^{-n^2\pi^2 t}\sin(n\pi x). \end{aligned}$$

Solution 61. First, we break the problem down into two

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) - 1 + x - x\cos(t), \\ w(0,t) = 0, \\ w(1,t) = 0, \\ w(x,0) = 0. \end{cases} \qquad \begin{cases} z_t(x,t) = z_{xx}(x,t), \\ z(0,t) = 0, \\ z(1,t) = 0, \\ z(x,0) = x(1-x). \end{cases}$$

The second problem is a basic heat equation, we have

$$z(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t}\sin(n\pi x),$$

where

$$c_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx = -\frac{4\left((-1)^n - 1\right)}{\pi^3 n^3}.$$

To find w, we write as

$$w(x,t) = \sum_{n=1}^{\infty} \alpha_n(t)\sin(n\pi x).$$

and we write $F(x,t)$ as

$$F(x,t) = \sum_{n=1}^{\infty} \beta_n(t)\sin(n\pi x),$$

where

$$\beta_n(t) = 2\int_0^1 F(x,t)\sin(n\pi x)\,dx = \frac{2\left((-1)^n\cos(t) - 1\right)}{\pi n}.$$

Now, α_n satisfies

$$\begin{cases} \alpha_n'(t) + n^2\pi^2\alpha_n(t) = \beta_n(t), \\ \alpha_n(0) = 0. \end{cases}$$

The solution (That you'd obtain via the method. of variation of the constant) is

$$\alpha_n(t) = \int_0^t \beta_n(r) e^{-n^2\pi^2(t-r)}\, dr$$

To compute this integral cleanly, we write $\beta_n(t)$ as $b_n\cos(t) + d_n$, then

$$\begin{aligned} \alpha_n(t) &= b_n \int_0^t \cos(t) e^{-n^2\pi^2(t-r)}\, dr + d_n \int_0^t e^{-n^2\pi^2(t-r)}\, dr \\ &= b_n \left(\frac{n^2\pi^2\cos(t) + \sin(t)}{n^4\pi^4 + 1} - \frac{n^2\pi^2}{n^4\pi^4+1} \right) + d_n \left(\frac{1 - e^{-n^2\pi^2 t}}{n^2\pi^2} \right), \\ &= \frac{2(-1)^n}{\pi n} \left(\frac{n^2\pi^2\cos(t) + \sin(t)}{n^4\pi^4 + 1} - \frac{n^2\pi^2}{n^4\pi^4+1} e^{-n^2\pi^2 t} \right) - \frac{2}{n\pi} \left(\frac{1 - e^{-n^2\pi^2 t}}{n^2\pi^2} \right), \\ &= \frac{2(-1)^n}{\pi n} \cdot \frac{n^2\pi^2\cos(t) + \sin(t)}{n^4\pi^4 + 1} - \frac{2}{n^3\pi^3} + \left(-\frac{2n\pi(-1)^n}{n^4\pi^4+1} + \frac{2}{n^3\pi^3} \right) e^{-n^2\pi^2 t} \end{aligned}$$

By combining w and z, we get

$$\begin{aligned} u(x,t) = &\sum_{n=1}^{\infty} \frac{2}{n\pi} \left[\frac{-1}{n^2\pi^2} + (-1)^n \frac{n^2\pi^2\cos(t) + \sin(t)}{n^4\pi^4 + 1} \right] \sin(n\pi x) \\ &+ \sum_{n=1}^{\infty} \left[\frac{6 - 4(-1)^n}{n^3\pi^3} - (-1)^n \frac{2n\pi}{n^4\pi^4 + 1} \right] e^{-n^2\pi^2 t} \sin(n\pi x). \end{aligned}$$

Note: When $t \to \infty$, the second sum goes to 0, we will be left with the first sum.

Exercise 62. Transform the following heat equation into two heat equations, both with homogeneous boundary conditions,

- One with a zero initial condition and a non-homogeneous term.
- One with a non-zero initial condition and **no** non-homogeneous terms.

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) \\ u_x(0,t) = f(t) \\ u_x(1,t) = g(t) \\ u(x,0) = h(x) \end{cases}$$

Solution 62. The most important part of this problem is finding ψ such that $u(x,t) = \psi(x,t) + v(x,t)$, where v satisfy the heat equation with homogeneous conditions. Notice that ψ depends on x and t, and

$$\psi_x(0,t) = f(t), \qquad \psi_x(1,t) = g(t).$$

Mimicking the Dirichlet BC case, we assume that

$$\psi(x,t) = a(x)f(t) + b(x)g(t)$$

to get

$$\begin{cases} a'(0) = 1 \\ a'(1) = 0 \end{cases}, \qquad \begin{cases} b'(0) = 0 \\ b'(1) = 1 \end{cases}$$

One way to proceed here is to assume that a and b are quadratic and find their coefficients, you should find

$$a(x) = x - \frac{x^2}{2}, \qquad b(x) = \frac{x^2}{2}.$$

Combined, they give $\psi(x,t) = \left(x - \frac{x^2}{2}\right) f(t) + \frac{x^2}{2} g(t)$. Plug it back into the equation of v to get $v_t(x,t) = v_{xx}(x,t) + \psi_{xx}(x,t) - \psi_t(x,t)$. Therefore,

$$\begin{cases} v_t(x,t) = v_{xx}(x,t) + g(t) - f(t) - \left(x - \frac{x^2}{2}\right) f'(t) - \frac{x^2}{2} g'(t) \\ v_x(0,t) = 0 \\ v_x(1,t) = 0 \\ v(x,0) = h(x) - \left(x - \frac{x^2}{2}\right) f(0) - \frac{x^2}{2} g(0) \end{cases}$$

Now, we come to easy part, where we break down this BVP into two by setting up $v(x,t) = w(x,t) + z(x,t)$ and

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) + g(t) - f(t) - \left(x - \frac{x^2}{2}\right) f'(t) - \frac{x^2}{2} g'(t) \\ w_x(0,t) = 0 \\ w_x(1,t) = 0 \\ w(x,0) = 0 \end{cases}$$

$$\begin{cases} z_t(x,t) = z_{xx}(x,t) \\ z_x(0,t) = 0 \\ z_x(1,t) = 0 \\ z(x,0) = h(x) - \left(x - \frac{x^2}{2}\right) f(0) - \frac{x^2}{2} g(0) \end{cases}$$

Exercise 63. Solve the 2D heat equation subject to the following boundary conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0$$
$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \qquad \left.\frac{\partial u}{\partial y}\right|_{y=1} = 0$$

and satisfies the following initial condition

$$u(x,y,0) = xy$$

Solution 63. Following the notes

$$u(x,y) = A_{0,0} + \sum_{n=1}^{\infty} A_{n,0}\cos(n\pi x)e^{-n^2\pi^2 t} + \sum_{m=1}^{\infty} A_{0,m}\cos(m\pi x)e^{-m^2\pi^2 t} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{n,m}\cos(n\pi x)\cos(m\pi y)e^{-(m^2+n^2)\pi^2 t},$$

where

$$A_{0,0} = \int_0^1\int_0^1 xy\,dxdy = \frac{1}{4},$$

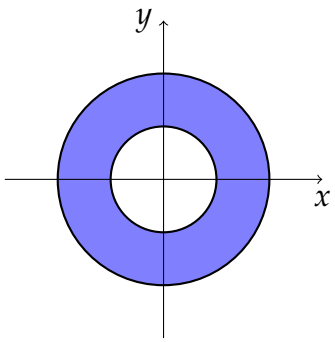
$$A_{n,0} = 2\int_0^1\int_0^1 xy\cos(n\pi x)\,dxdy = \frac{(-1)^n - 1}{n^2\pi^2},$$

$$A_{0,m} = 2\int_0^1\int_0^1 xy\cos(m\pi y)\,dxdy = \frac{(-1)^m - 1}{m^2\pi^2},$$

$$A_{n,m} = 4\int_0^1\int_0^1 xy\cos(n\pi x)\cos(m\pi y)\,dxdy = \frac{4\left((-1)^n - 1\right)\left((-1)^m - 1\right)}{m^2n^2\pi^4}.$$

Exercise 64. Find the steady state temperature $u(r,\theta)$ on the annular plate $1 < r < 2$ subject to

$$u(1,\theta) = \theta(2\pi - \theta), \qquad u(2,\theta) = 1$$



Hint: Break it down into two problems.

Solution 64. We split the problem into two $u = w + z$, where w satisfies

$$w(1,\theta) = f(\theta) = \theta(2\pi - \theta), \qquad w(2,\theta) = 0.$$

and z satisfies

$$z(1,\theta) = 0, \qquad z(2,\theta) = g(\theta) = 1.$$

- Let us start with w: We recall Formula A

$$w(r,\theta) = \alpha_1 + \alpha_2 \ln(r) + \sum_{n=1}^{\infty} \left(c_n r^n + d_n r^{-n}\right) \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right).$$

We start by enforcing the condition $w(2,\theta) = 0$:

$$w(2,\theta) = 0 = \alpha_1 + \alpha_2 \ln(2) + \sum_{n=1}^{\infty} \left(c_n 2^n + d_n 2^{-n}\right) \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right).$$

Here, we can see that

$$\alpha_1 = -\alpha_2 \ln(2), \qquad d_n = -4^n c_n$$

As usual, we will take $c_n 2^n$ to be 1. Therefore

$$w(r,\theta) = \alpha_2 \ln\left(\frac{r}{2}\right) + \sum_{n=1}^{\infty} \left(\left(\frac{r}{2}\right)^n - \left(\frac{2}{r}\right)^n\right) \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right).$$

Now, we plug in 1 for r to enforce the boundary condition $u(1,\theta) = \theta(2\pi - \theta)$:

$$\theta(2\pi - \theta) = -\alpha_2 \ln(2) + \sum_{n=1}^{\infty} \left(2^{-n} - 2^n\right) \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)$$

Using the classical Fourier series formulas, we get

$$-\alpha_2 \ln(2) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta \longrightarrow \alpha_2 = -\frac{2\pi^2}{3\ln(2)}.$$

$$\left(2^{-n} - 2^n\right) A_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\, d\theta \longrightarrow A_n = \frac{1}{2^{-n}-2^n}\cdot\frac{-4}{n^2}.$$

$$\left(2^{-n} - 2^n\right) B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\, d\theta \longrightarrow B_n = 0.$$

Finally, we have

$$w(r,\theta) = \frac{-2\pi^2}{3\ln(2)}\ln\left(\frac{r}{2}\right) + \sum_{n=1}^{\infty}\left(\left(\frac{r}{2}\right)^n - \left(\frac{2}{r}\right)^n\right)\left(\frac{1}{2^{-n}-2^n}\cdot\frac{-4}{n^2}\right)\cos(n\theta).$$

- We solve for z similarly: we start from formula A

$$z(r,\theta) = \alpha_1 + \alpha_2\ln(r) + \sum_{n=1}^{\infty}\left(c_n r^n + d_n r^{-n}\right)\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right).$$

We plug in 1 for r to get

$$z(1,\theta) = \alpha_1 + \sum_{n=1}^{\infty}\left(c_n + d_n\right)\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)$$

so $\alpha_1 = 0$ and $c_n = -d_n$, we choose $c_n = 1$ to get

$$z(r,\theta) = \alpha_2\ln(r) + \sum_{n=1}^{\infty}\left(r^n - r^{-n}\right)\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right).$$

Now, we plug 2 for r to get

$$g(\theta) = \alpha_2\ln(2) + \sum_{n=1}^{\infty}\left(2^n - 2^{-n}\right)\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right).$$

and we use Fourier series formulas to obtain:

$$\alpha_2\ln(2) = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)d\theta \longrightarrow \alpha_2 = \frac{1}{\ln(2)}.$$

$$\left(2^n - 2^{-n}\right) A_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos(n\theta)\, d\theta \longrightarrow A_n = 0.$$

$$\left(2^n - 2^{-n}\right) B_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin(n\theta)\, d\theta \longrightarrow B_n = 0.$$

Finally, we have

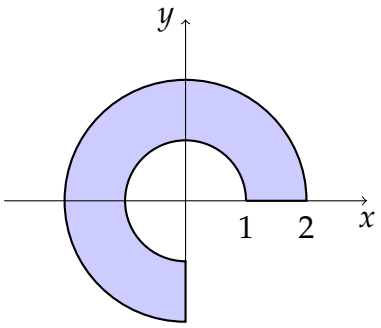
$$z(r,\theta) = \frac{1}{\ln(2)}\ln(r).$$

At the end, we sum z and w to get the final answer:

$$u(r,\theta) = \frac{\ln(r)}{\ln(2)} + \frac{-2\pi^2}{3\ln(2)} \ln\left(\frac{r}{2}\right) + \sum_{n=1}^{\infty} \left(\left(\frac{r}{2}\right)^n - \left(\frac{2}{r}\right)^n \right) \left(\frac{1}{2^{-n} - 2^n} \cdot \frac{-4}{n^2} \right) \cos(n\theta).$$

Exercise 65. Find the steady state temperature $u(r,\theta)$ on the region $1 < r < 2,\ 0 < \theta < 3\pi/2$ subject to

$$u(1,\theta) = \theta, \qquad u(2,\theta) = \theta(2\pi - 3\theta), \qquad u(r,0) = u(r,\frac{3\pi}{2}) = 0.$$



Solution 65. The SL problem is

$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0, \\ \Theta(0) = \Theta\left(\frac{3\pi}{2}\right) = 0. \end{cases}$$

The eigenfunctions are $\Theta_n(\theta) = \sin\left(\frac{2n\theta}{3}\right)$. Therefore, the solution u can be written as

$$u(r,\theta) = \sum_{n=1}^{\infty} \left(c_n r^{2n/3} + d_n r^{-2n/3}\right) \sin\left(\frac{2n\theta}{3}\right).$$

Now, we just need to make sure that u satisfies the boundary conditions. My preferred way is to split the problem into two problems

$$u = w + z, \qquad \begin{cases} w(1,\theta) = 0, \\ w(2,\theta) = \theta(2\pi - 3\pi) \end{cases}, \qquad \begin{cases} z(1,\theta) = \theta, \\ z(2,\theta) = 0. \end{cases}$$

- **Solving for** w: We start by enforcing the condition $w(1,\theta) = 0$:

$$0 = \sum_{n=1}^{\infty} (c_n + d_n) \sin\left(\frac{2n\theta}{3}\right) \longrightarrow d_n = -c_n.$$

Now, we have

$$w(r,\theta) = \sum_{n=1}^{\infty} c_n \left(r^{2n/3} - r^{-2n/3}\right) \sin\left(\frac{2n\theta}{3}\right).$$

Plug in 2 for r to get

$$\theta(2\pi - 3\theta) = \sum_{n=1}^{\infty} c_n \left(2^{2n/3} - 2^{-2n/3}\right) \sin\left(\frac{2n\theta}{3}\right).$$

Using FS formulas:

$$c_n\left(2^{2n/3}-2^{-2n/3}\right)=\frac{2}{\frac{3\pi}{2}}\int_0^{\frac{3\pi}{2}}\theta(2\pi-3\theta)\sin\left(\frac{2n\theta}{3}\right)\,d\theta$$

Plug it into Mathematica, to get

$$c_n=\frac{3\left(5\pi^2(-1)^n n^2-18(-1)^n+18\right)}{2\pi n^3\left(2^{2n/3}-2^{-2n/3}\right)}.$$

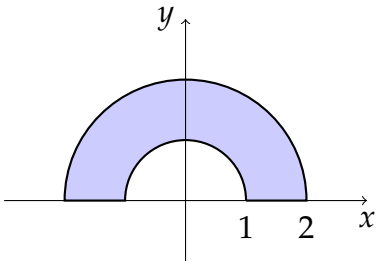
Finally, we have

$$w(r,\theta)=\sum_{n=1}^{\infty}\frac{3\left(5\pi^2(-1)^n n^2-18(-1)^n+18\right)}{2\pi n^3\left(2^{2n/3}-2^{-2n/3}\right)}\left(r^{2n/3}-r^{-2n/3}\right)\sin\left(\frac{2n\theta}{3}\right)$$

- **Solving for** z: This can be done in a similar way. I will omit the details here but feel free to ask me questions if you have any issues.

Exercise 66. Find the steady state temperature $u(r,\theta)$ on the semi-annular plate $1 < r < 2,\ 0 < \theta < \pi$ subject to

$$u(1,\theta) = 0, \qquad u(2,\theta) = \theta(2\pi - \theta), \qquad u(r,0) = u(r,\pi) = 0.$$



Solution 66. The solution can be expressed as

$$u(r,\theta) = \sum_{n=1}^{\infty} (c_n r^n + d_n r^{-n}) \sin(n\theta).$$

Next, we enforce the BCs on c_n and d_n:

$$c_n + d_n = 0 \Longrightarrow d_n = -c_n.$$

We go back to the series representation of z:

$$u(r,\theta) = \sum_{n=1}^{\infty} c_n (r^n - r^{-n}) \sin(n\theta).$$

To verify the last boundary condition, we set $r = 2$:

$$\theta(2\pi - \theta) = \sum_{n=1}^{\infty} c_n (2^n - 2^{-n}) \sin(n\theta).$$

Therefore,

$$c_n = \frac{2}{(2^n - 2^{-n})\pi} \int_0^{\pi} \theta(2\pi - \theta) \sin(n\theta)\, d\theta = \frac{2\left((-1)^{n+1}\left(\pi^2 n^2 + 2\right) + 2\right)}{\pi n^3 (2^n - 2^{-n})}.$$

Finally, we have

$$u(r,\theta) = \sum_{n=1}^{\infty} \left(\frac{r^n - r^{-n}}{2^n - 2^{-n}}\right) \frac{2\left((-1)^{n+1}\left(\pi^2 n^2 + 2\right) + 2\right)}{\pi n^3 (2^n - 2^{-n})} \sin(n\theta).$$

Exercise 67. Find the steady state temperature $u(r,\theta)$ on the disk $r < 1$ subject to

$$u(1,\theta) = \theta(2\pi - \theta).$$

Solution 67. The solution u can be written as

$$u(r,\theta) = \alpha_1 + \sum_{n=1}^{\infty} r^n \left(A_n \cos(\theta) + B_n \sin(\theta)\right),$$

where

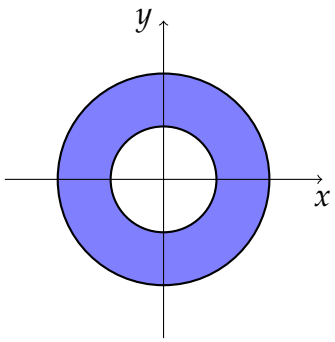
$$\alpha_1 = \frac{1}{2\pi}\int_0^{2\pi} \theta(2\pi - \theta)\, d\theta = \frac{2\pi^2}{3}$$

$$A_n = \frac{1}{\pi}\int_0^{2\pi} \cos(n\theta)\theta(2\pi - \theta)\, d\theta = \frac{-4}{n^2}$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} \sin(n\theta)\theta(2\pi - \theta)\, d\theta = 0$$

Exercise 68. Find the steady state temperature $u(r,\theta)$ on the annular plate $1 < r < 2$ subject to

$$u(1,\theta) = \theta^2(2\pi-\theta)^2, \qquad u(2,\theta) = \sin(\theta).$$



Hint: Break it down into two problems.

Solution 68. We break u down into $u = z + w$, where

$$\begin{cases} z(1,\theta) = \theta^2(2\pi-\theta)^2, \\ z(2,\theta) = 0 \end{cases}, \qquad \begin{cases} w(1,\theta) = 0, \\ w(2,\theta) = \theta(2\pi-\theta). \end{cases}$$

The steps for finding z and w are almost identical:

1. Finding z: We write z as

$$z(r,\theta) = \alpha_1 + \alpha_2 \ln(r) + \sum_{n=1}^{\infty} \left(c_n r^n + d_n r^{-n}\right)\left(A_n \cos(n\theta) + B_n \sin(n\theta)\right).$$

Now, we enforce the boundary conditions on α_1, α_2:

$$\alpha_1 + \alpha_2 \ln(2) = 0 \Longrightarrow \alpha_1 = -\alpha_2 \ln(2)$$

Next, we enforce the BCs on c_n and d_n:

$$c_n 2^n + d_n 2^{-n} = 0 \Longrightarrow d_n = -2^{2n} c_n.$$

We go back to the series representation of z:

$$z(r,\theta) = \alpha_2\left(\ln(r) - \ln(2)\right) + \sum_{n=1}^{\infty} c_n\left(r^n - 2^{2n} r^{-n}\right)\left(A_n \cos(n\theta) + B_n \sin(n\theta)\right).$$

We can see that we can drop c_n since it is multiplied by A_n and B_n

$$z(r,\theta) = \alpha_2\left(\ln(r) - \ln(2)\right) + \sum_{n=1}^{\infty} \left(r^n - 2^{2n} r^{-n}\right)\left(A_n \cos(n\theta) + B_n \sin(n\theta)\right).$$

To verify the last boundary condition, we set $r = 1$:

$$\theta^2(2\pi-\theta)^2 = -\ln(2)\alpha_2 + \sum_{n=1}^{\infty}(1-2^{2n})\left(A_n\cos(n\theta)+B_n\sin(n\theta)\right)$$

Therefore,

$$\alpha_2 = \frac{-1}{2\pi\ln(2)}\int_0^{2\pi}\theta^2(2\pi-\theta)^2\,d\theta = \frac{-1}{\ln(2)} = \frac{-8\pi^4}{15\ln(2)}.$$

$$A_n = \frac{1}{(1-2^{2n})\pi}\int_0^{2\pi}\theta^2(2\pi-\theta)^2\cos(n\theta)\,d\theta = \frac{-48}{(1-2^{2n})n^4}.$$

$$B_n = \frac{1}{(1-2^{2n})\pi}\int_0^{2\pi}\theta^2(2\pi-\theta)^2\sin(n\theta)\,d\theta = 0.$$

Finally, we have

$$z(r,\theta) = \frac{-8\pi^4}{15\ln(2)}\left(\ln(r)-\ln(2)\right) + \sum_{n=1}^{\infty}\frac{-48(r^n-2^{2n}r^{-n})}{(1-2^{2n})n^4}\cos(n\theta).$$

2. Finding w: We write w as

$$w(r,\theta) = \alpha_1 + \alpha_2\ln(r) + \sum_{n=1}^{\infty}\left(c_nr^n + d_nr^{-n}\right)\left(A_n\cos(n\theta)+B_n\sin(n\theta)\right).$$

Now, we enforce the boundary conditions on α_1, α_2:

$$\alpha_1 + \alpha_2\ln(1) = 0 \Longrightarrow \alpha_1 = 0$$

Next, we enforce the BCs on c_n and d_n:

$$c_n + d_n = 0 \Longrightarrow d_n = -c_n.$$

We go back to the series representation of w:

$$w(r,\theta) = \alpha_2\ln(r) + \sum_{n=1}^{\infty}c_n(r^n-r^{-n})\left(A_n\cos(n\theta)+B_n\sin(n\theta)\right).$$

We can see that we can drop c_n since it is multiplied by A_n and B_n

$$w(r,\theta) = \alpha_2\ln(r) + \sum_{n=1}^{\infty}(r^n-r^{-n})\left(A_n\cos(n\theta)+B_n\sin(n\theta)\right).$$

To verify the last boundary condition, we set $r=2$:

$$\sin(\theta) = \ln(2)\alpha_2 + \sum_{n=1}^{\infty}(2^n-2^{-n})\left(A_n\cos(n\theta)+B_n\sin(n\theta)\right)$$

Therefore,

$$\alpha_2 = \frac{1}{2\pi \ln(2)} \int_0^{2\pi} \sin(\theta)\, d\theta = 0.$$

When computing A_n and B_n, we will have to discuss the case $n = 1$ separately:

$$A_n = \frac{1}{(2^n - 2^{-n})\pi} \int_0^{2\pi} \sin(\theta)\cos(n\theta)\, d\theta = \begin{cases} 0, & \text{if } n > 1, \\ 0, & \text{if } n = 1. \end{cases}.$$

$$B_n = \frac{1}{(2^n - 2^{-n})\pi} \int_0^{2\pi} \sin(\theta)\sin(n\theta)\, d\theta = \begin{cases} 0, & \text{if } n > 1, \\ \frac{1}{2-2^{-1}} & \text{if } n = 1. \end{cases}.$$

Finally, we have

$$w(r,\theta) = \frac{(r - r^{-1})}{2 - 2^{-1}} \sin(\theta) = \frac{2\left(r - \frac{1}{r}\right)}{3} \sin(\theta).$$

By adding z and w, we get

$$\boxed{u(r,\theta) = \frac{2\left(r - \frac{1}{r}\right)}{3} \sin(\theta) + \frac{-8\pi^4}{15\ln(2)} (\ln(r) - \ln(2)) + \sum_{n=1}^{\infty} \frac{-48(r^n - 2^{2n} r^{-n})}{(1 - 2^{2n})n^4} \cos(n\theta).}$$

Technical notes:

1. The function $\theta^2(2\pi - \theta)^2$ was chosen such that the Fourier series converges fast, you only need around 10 terms in the sum to get a good approximation. I leave it up to you to figure out why the FS converges fast.
2. There are ways to compute the FS of w without doing any integrals. Also, there is a way to see that B_n coefficient of z is zero without computing the integral.

Exercise 69. Use Laplace transform to solve the following wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \quad x > 0, \quad t > 0 \\ \\ u(0,t) = 0, \quad \lim\limits_{x\to\infty} u(x,t) = 0, \quad t > 0 \\ \\ u(x,0) = 0, \quad \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = xe^{-4x}, \quad x > 0 \end{cases}$$

Solution 69. Apply LT with respect to t to get

$$\begin{cases} \dfrac{d^2}{dx^2}U(x,s) = s^2 U(x,s) - xe^{-4x}, \\ \\ U(0,s) = 0, \quad \lim\limits_{x\to\infty} U(x,s) = 0. \end{cases}$$

The homogeneous (also called complementary) solution is $U_h(x,s) = c_1 e^{sx} + c_2 e^{-sx}$ (we can use the limit condition here but we will postpone until the end). The particular solution is of the form

$$U_p(x,s) = (ax+b)e^{-4x}.$$

Plug it back into the differential equation

$$\Big(-a(s^2-16) - b(s^2-16) - 8a\Big) = -xe^{-4x}.$$

By comparing the coefficients of x and 1 on each side, we get

$$\begin{cases} a(s^2-16) & = 1 \\ -b(s^2-16) - 8a & = 0 \end{cases}$$

This is a triangular system, we can solve for a, then b, then c to get

$$a = \frac{1}{s^2-16}, \qquad b = -\frac{8}{(s^2-16)^2},$$

Now, we combine the particular solution and the homogeneous solution to get

$$U(x,s) = c_1 e^{sx} + c_2 e^{-sx} + \left(\frac{-8}{(s^2-16)^2} + \frac{1}{s^2-16}x\right)e^{-4x}.$$

The condition $\lim\limits_{x\to\infty} U(x,s) = 0$ is equivalent to $c_1 = 0$, this makes U a little bit simpler

$$U(x,s) = c_2 e^{-sx} + \left(\frac{-8}{(s^2-16)^2} + \frac{1}{s^2-16}x\right)e^{-4x}.$$

The condition $U(0,s) = 0$ is equivalent to $c_2 = \dfrac{8}{(s^2-16)^2}$. Therefore.

$$U(x,s) = \frac{8}{(s^2-16)^2}e^{-sx} + \left(\frac{-8}{(s^2-16)^2} + \frac{1}{s^2-16}x\right)e^{-4x}.$$

There are two ILTs that we need to compute, let us start with the easiest

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-16}\right\} = \boxed{\sinh(4t)}. \qquad \text{(From the table).}$$

The second one a little trickier but we can use the convolution theorem to evaluate it:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{8}{(s^2-16)^2}\right\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{4}{s^2-16}\cdot\frac{4}{s^2-16}\right\} \\ &= \frac{1}{2}\int_0^t \sinh(4r)\sinh(4(t-r))dr = \boxed{\frac{1}{16}(4t\cosh(4t)-\sinh(4t))}.\end{aligned}$$

(There is another to do it using the derivative of transforms theorem).
Finally, we can put everything together to get

$$\boxed{u(x,t)=\left(\tfrac{1}{16}(4(t-x)\cosh(4(t-x))-\sinh(4(t-x))\right)\mathscr{U}(t-x)+\left(\sinh(4t)x+\tfrac{1}{16}(4t\cosh(4t)-\sinh(4t))\right)e^{-4x}.}$$

Exercise 70. Represent the solution of the following heat equation as an improper integral using Fourier transform.

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = g(x), \end{cases}$$

where

$$g(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| > 1. \end{cases}$$

Solution 70. Since the domain of x is $(-\infty, \infty)$, we need to apply the regular (exponential) Fourier transform:

$$U_t(\alpha, t) = -\alpha^2 U(\alpha, t)$$

The solution to this homogeneous ODE is

$$U(\alpha, t) = c_0 e^{-\alpha^2 t}$$

We also have $U(\alpha, 0) = G(\alpha)$, where G is the Fourier transform of g:

$$G(\alpha) = \int_{-\infty}^{\infty} g(x) e^{-i\alpha x}\, dx = \int_{-1}^{1} e^{i\alpha x}\, dx$$

The last integral is pretty straightforward:

$$G(\alpha) = \left[\frac{e^{i\alpha x}}{i\alpha}\right]_{\alpha=-1}^{1} = \frac{2}{\alpha} \cdot \frac{e^{i\alpha} - e^{-i\alpha}}{2i} = \frac{2\sin(\alpha)}{\alpha}.$$

Using $U(\alpha, 0) = G(\alpha)$, we get that $c_0 = G(\alpha)$. Now, we apply the inverse Fourier transform:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin(\alpha)}{\alpha} e^{-\alpha^2 t} e^{-i\alpha x}\, d\alpha$$

Exercise 71. Find the Fourier integral representation of the following function

$$f(x) = \begin{cases} 0, & x < -1 \\ 2, & -1 < x < 1 \\ 0, & x > 1 \end{cases}.$$

Solution 71. Since f is even, we have $B(\alpha) = 0$ and

$$A(\alpha) = 2\int_0^\infty f(x)\cos(\alpha x)\,dx = 4\int_0^1 \cos(\alpha x)\,dx = \frac{4\sin(\alpha)}{\alpha}.$$

Therefore,

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{4\sin(\alpha)}{\alpha}\cos(\alpha x)\,d\alpha.$$

Exercise 72. Find the steady state temperature $u(x,y)$ on the domain $\{(x,y) \mid x \geq 0, y \leq 0\}$ (The fourth quadrant) subject to $u_y(x,0) = e^{-x}$ and $u(0,y) = 0$.

Solution 72. Notice that x is in $(0,\infty)$ and y is in $(-\infty,0)$ which means that the regular FT cannot be applied. It is preferable to apply a transform with respect to x since it is in $(0,\infty)$. If you wish to apply a transform with respect to y, you have to be very careful since our formulas were developed for integrals on $(0,\infty)$. Since $u(0,y)$ is specified (a Dirichlet BC), we need to use the sine transform:

$$U(\alpha,y) = \mathcal{F}_s\{u(x,y)\}.$$

We have $u_{xx}(x,y) + u_{yy}(x,y) =$, rearrange it to get

$$u_{xx}(x,y) = -u_{yy}(x,y)$$

Apply the sine transform and use $u(0,y) = 0$:

$$-\alpha^2 U(\alpha,y) = -U_{yy}(\alpha,y)$$

The solution to this ODE is simple

$$U(\alpha,y) = Ae^{\alpha y} + Be^{-\alpha y}$$

We need U to be bounded as $\alpha \to \infty$ in order to define the inverse transform, so $B = 0$ (because y is negative). We also have

$$U(\alpha,0) = \alpha A = \mathcal{F}_s\{e^{-x}\} = \frac{\alpha}{\alpha^2+1}. \qquad \text{(From the notes)}$$

So,

$$U(\alpha,y) = \frac{1}{\alpha^2+1}e^{\alpha y}.$$

Finally, we apply the inverse sine transform:

$$\boxed{u(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\alpha^2+1}e^{\alpha y}\sin(\alpha x)\,d\alpha.}$$

Exercise 73. Find the Fourier integral representation of the following function

$$f(x) = \begin{cases} 0, & x < \pi \\ 4, & \pi < x < 2\pi \\ 0, & x > 2\pi \end{cases}.$$

Solution 73. We have

$$A(\alpha) = \int_{-\infty}^{\infty} f(z)\cos(\alpha z)\,dz = \int_{\pi}^{2\pi} 4\cos(\alpha z)\,dz = \frac{4(\sin(2\pi\alpha) - \sin(\pi\alpha))}{\alpha}.$$

Similarly,

$$B(\alpha) = \int_{-\infty}^{\infty} f(z)\sin(\alpha z)\,dz = \int_{\pi}^{2\pi} 4\sin(\alpha z)\,dz = \frac{4(\cos(\pi\alpha) - \cos(2\pi\alpha))}{\alpha}.$$

Therefore, the Fourier integral representation of f is

$$\boxed{f(x) = \frac{1}{\pi}\int_0^{\infty} \frac{4(\sin(2\pi\alpha) - \sin(\pi\alpha))}{\alpha}\cos(\alpha x) + \frac{4(\cos(\pi\alpha) - \cos(2\pi\alpha))}{\alpha}\sin(\alpha x)\,d\alpha.}$$

Note: There is actually a nicer way to write the integrand:

$$f(x) = \frac{8}{\pi}\int_0^{\infty} \frac{\sin\left(\frac{\pi\alpha}{2}\right)\cos\left(\alpha\left(\frac{3\pi}{2} - x\right)\right)}{\alpha}\,d\alpha$$